This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

NYPL RESEARCH LIBRARIES
3 3433 06633339 8

PBM

ELEMENTS

OF THE

KINEMATICS OF A POINT

AND THE

RATIONAL MECHANICS OF A PARTICLE

BY

G. O. JAMES, PH.D.

Instructor in Mathematics and Astronomy
Washington University, St. Louis

FIRST EDITION

FIRST THOUSAND

1745

NEW YORK
JOHN WILEY & SONS
LONDON: CHAPMAN & HALL, LIMITED
1905

York Public Library
WEBSTER BRANCH,
538 East 76th Street.
CENTRAL COLLECTION
CIRCULATING DEPARTMENT

THE NEW YORK
PUBLIC LIBRARY
240210B
ASTOR, LENOX AND
TILDEN FOUNDATIONS
R 1943 L

Copyright, 1905,
BY
G. O. JAMES

ROBERT DRUMMOND, PRINTER, NEW YORK

531
J3

48664
20227
c.
wb

INTRODUCTION

THIS book is intended for those who expect to continue the study of mechanics beyond an elementary course, and is meant to serve as an introduction to advanced treatises. For this reason special attention has been given to the principles and order of presentation, while the applications have been left almost entirely aside. No attempt has been made to avoid such mathematical terms and formulæ as seemed necessary, but those problems requiring a knowledge beyond the calculus and elementary differential equations have either been entirely omitted, or approximate solutions only have been given. Foucault's pendulum has been treated in this way. Especial attention has been given to relative motion and to motion on the Earth's surface, and to obtain a proper orientation in the subject the problems chosen have been made as general as possible.

G. O. JAMES.

WASHINGTON UNIVERSITY, May 2, 1904.

CONTENTS

PART I

KINEMATICS

CHAPTER I

THEORY OF VECTORS

ARTICLE PAGE

1. Vectors 1
2. Equipollence of two vectors and notation 2
3. Geometric sum 3
4. Analytic expression of geometric sum 5
5. Geometric difference 8
6. Analytic expression of geometric difference 9
7. Decomposition of a vector 9
8. Analytic expression of components 10
9. Projection of a vector on a line 11
10. Analytic expression of geometric sum referred to axes 13
11. Geometric derivatives 14
12. Projection of the geometric derivative of a vector on a plane and on an axis 16
13. Projection of the geometric derivative of a vector on the vector itself 19
14. Applications 19

CHAPTER II

KINEMATICS OF A POINT. GENERAL THEOREMS

15. Definitions. Rest and motion 22
16. Time 23

ARTICLE PAGE
17. Motion of a point. Path. 23
18. Equations of motion. 23
19. Rectilinear motion. 24
20. Units of time and of length. 25
21. Case where it is necessary to specify the units. 25
22. Homogeneous equations. 26
23. Change of units. 26
24. Uniform rectilinear motion. Velocity 28
25. Equation of uniform motion. 29
26. Accelerated retilinear motion. 30
27. Numerical examples. 30
28. Knots and nautical miles. 31
29. Equation of accelerated motion in terms of the mean velocity. . . 32
30. Acceleration. 32
31 Equation of uniformly accelerated motion. 33
32. Discussion of the motion by means of the equation of motion 34
33. Case where the displacement is taken as the independent variable 35

CHAPTER III

APPLICATIONS TO ORDINARY RECTILINEAR MOTIONS

A. Uniform Motion

34. Properties of uniform motion. 37
35. Problem. 38

B. Uniformly Accelerated Motion

36. Properties of uniformly accelerated motion. 39
37. Remark. 42
38. Equations obtained by taking the space described as independent variable. 42

C. Periodic Motions

39. Equations of periodic motion. 43
40. Equations of harmonic motion. 44
41. Amplitude, elongation, frequency, argument, phase. 45
42. Velocity and acceleration. 47
43. Discussion of the motion. 47
44. Influence of the phase. 49

CHAPTER IV

RELATIVE MOTION ALONG A LINE

ARTICLE PAGE
45. Relative motion. 51
46. Equation of relative motion. 52
47. Relative velocity and acceleration. 52
48. Apparent motion. 53
49. Convective motion. 54
50. Simultaneous motions. 55
51. Composition of harmonic motions of same period. 56
52. Representation of the amplitude and phase by a vector. 57
53. Equation of the resultant motion. 58
54. Vibrations of different periods. 59

CHAPTER V

VELOCITY AND ACCELERATION IN CURVILINEAR MOTION

A. Velocity

55. Displacement. 61
56. Velocity. 62
57. Algebraic value of the velocity. 63
58. Projection of the velocity on the displacement. 64
59. Geometric representation of the path. 65
60. Projection of the velocity on any axis. 65

B. Acceleration

61. Acceleration. Hodograph. 66
62. Projection of the acceleration on any axis. 67
63. Projection of the acceleration on the tangent and principal normal. 67
64. Projection of the acceleration on the binormal. 68
65. Condition that the acceleration be constantly tangential or normal to the path. Condition that the acceleration be constantly zero. 69
66. Composition of motions along the same path. 69

CHAPTER VI

ANGULAR AND AREAL MOTION. EQUATIONS AND GENERAL THEOREMS

ARTICLE PAGE
67. Angular displacement........ 70
68. Angular velocity 71
69. Relation between linear and angular velocities........ 72
70. Angular acceleration........ 72
71. Relation between linear and angular accelerations........ 73
72. Areal velocity........ 74
73. Extension of theorems already found........ 76
74. Equations of motion. Units of time and displacement. Homogeneity........ 76
75. Equation of uniform motion........ 77
76. Equation of uniformly accelerated motion........ 78
77. Equation of motion when angular acceleration is not constant 78
78. Properties of uniformly accelerated angular motion........ 78
79. Periodic angular motion. Harmonic motion........ 79
80. Relative angular motion........ 80
81. Composition of harmonic angular motions of same period.... 82

CHAPTER VII

MOTION REFERRED TO COORDINATE AXES

82. Important remark........ 83
83. Projection of the motion on a plane and on an axis........ 83
84. Projection of the velocity and acceleration........ 84
85. Equations of motion referred to the coordinate axes........ 85
86. Equations of the projected motion on the coordinate planes..... 86
87. Projection of the path........ 86
88. Equations of the projected motion on the coordinate axes....... 87
89. Projections of the velocity........ 87
90. Equations of motion of the point which describes the hodograph 88
91. Projections of the acceleration on the coordinate axes........ 88
92. Résumé........ 89
93. Most general motion in which the projected motions are uniformly accelerated........ 90
94. Motion in which the projected motions are harmonic........ 91
95. Rectilinear harmonic motion considered as the projection of uniform circular motion........ 93

CHAPTER VIII

RELATIVE MOTION. MOVING AXES

ARTICLE PAGE
96. Fixed and moving axes. 95
97. Equations of absolute and relative motion. 96
98. Statement of the problem. 96
99. Motion of the relative axes. 96
100. Solution of the problem 97

PART II

MECHANICS

CHAPTER IX

MECHANICS OF A FREE PARTICLE

101. Material point or particle. 101
102. Purpose of this book. 102
103. Problem of mechanics. 103
104. The rôle of observation and experiment. 104
105. Principles of mechanics. 104
106. The absolute axes. Isolated particle. 104
107. First principle of mechanics. 105
108. Meaning of the first principle. 105
109. Field of force. 105
110. Uniform field of force. 106
111. Constant field of force. 106
112. Superposition of fields of force. 106
113. Second principle of mechanics. 106
114. Meaning of the second principle. 106
115. Reaction of the particle on the field. 107
116. Third principle of mechanics. 107
117. Meaning of third principle. 108
118. Properties of the coefficients ρ_{ij}. 108
119. Definition of mass. 109
120. Force. 110
121. Observations on the notion of force. 110
122. Composition of forces. Resultant. 111
123. Decomposition of forces. Components. 112
124. Equilibrium of a free particle. 113

ARTICLE PAGE
125. Force acting on a particle at rest 114
126. New definition of direction of force 114
127. Tangential and normal force 115
128. Resistance 115

CHAPTER X

THE UNITS OF MECHANICS

129. The units of kinematics 116
130. The units of mechanics 116
131. The three fundamental units. Derived units 116
132. Systems of units 117
133. C.G.S. system 117
134. M.K.S. system 117
135. English system 117
136. Units of astronomy 117
137. Unit of force 117
138. Remark 118

CHAPTER XI

GENERAL NOTIONS ON THE FORCES OF NATURE

139. Principle of the equality of action and reaction 119
140. Analytic expression of the common force between two free particles 120
141. Central forces 121
142. Universal gravitation 121
143. Note 122

CHAPTER XII

DETERMINATION OF THE LAW OF FORCE FROM THE MOTION PRODUCED

144. Finite and differential equations of motion 123
145. The two general problems of the mechanics of a particle in terms of force 124
146. The first problem 125
147. The different forces capable of producing a given rectilinear motion 126
148. The force producing uniformly accelerated motion 127

ARTICLE PAGE
149. Law of force porducing harmonic motion.................. 127
150. Law of force producing harmonic motion with coefficient of extinction.. 127
151. On the resistance producing extinction...................... 128
152. Law of force producing uniform curvilinear motion......... 129

CHAPTER XIII

MOTION PRODUCED BY FORCES OBEYING KNOWN LAWS

153. Statement of the problem.................................. 131
154. Attraction toward a fixed centre proportional to the distance 133

CHAPTER XIV

RELATIVE MOTION

155. Relative axes.. 141
156. Relative force... 141
157. Problems of relative motion............................... 141
158. First problem.. 142
159. Second problem.. 143

CHAPTER XV

MOTION OF A FREE PARTICLE ON EARTH'S SURFACE

160. The relative axes.. 145
161. Differential equations of motion.......................... 146
162. First approximation....................................... 146
163. Second approximation...................................... 147

CHAPTER XVI

CONSTRAINED MOTION OF A PARTICLE. GENERAL CONSIDERATIONS

164. Constrained particle....................................... 149
165. Mathematical expression of the conditions of constraint........ 149
166. Variable constraints....................................... 150
167. Smooth constraints.. 150
168. Forces of constraint...................................... 150
169. Motion of a constrained particle.......................... 151

ARTICLE PAGE
170. Direction of F_r. 151
171. Magnitude of F_r. 152
172 Particle at rest on a smooth surface. 152
173. Curve of constraint. 152
174. Particle at rest on a smooth curve. 153
175. Conditions of equilibrium. 153

CHAPTER XVII

CONSTRAINED MOTION ON EARTH'S SURFACE

176. Variation of g with latitude. 154
177. Particle falling on smooth curve. 156
178. Oscillations of a simple pendulum. 159
179. Oscillations of small amplitude. 161
180. Velocity in oscillations of any amplitude. 162
181. Tension in the string. 162
182. Foucault's pendulum. 163
183. Approximate solution. Particular case. General case. Foucault's case. Theorem of Chevilliet. 166–170

4111

PROPERTY OF
THE CITY OF NEW YORK.

ELEMENTS OF KINEMATICS AND MECHANICS

CHAPTER I

THEORY OF VECTORS

1. Vectors.—2. Equipollence of two vectors and notation.—3. Geometric sum.—4. Analytic expression of geometric sum.—5. Geometric difference.—6. Analytic expression of geometric difference.—7. Decomposition of a vector.—8. Analytic expression of the components.—9. Projection of a vector on a line.—10. Analytic expression of geometric sum referred to axes.—11. Geometric derivatives.—12. Projection of the geometric derivative of a vector on a plane and on an axis.—13. Projection of the geometric derivative of a vector on the vector itself.—14. Applications.

1. Vectors.—Many of the quantities of mechanics can be conveniently represented by segments of straight lines having a definite direction and length. Such a segment is termed a *vector*.

A *vector* is a segment AP (fig. 1) of a straight line having an origin A and an extremity P. It is defined by the following elements:

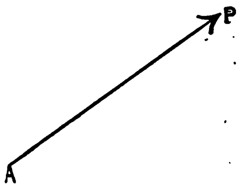


FIG. 1.

1° Its origin A;

2° Its length AP;

3° Its direction, which is by definition the direction of the line AP *from A toward P.*

The vector is always written by placing the letter representing the origin first. Thus the vector AP has its direction from A to P, while the vector PA has its direction from P toward A.

This is expressed by writing

$$PA = -AP \qquad [1]$$

Ordinarily, a vector AP is denoted by the single letter P and an arrow placed at P to denote the direction.

2. Equipollence of two vectors and notation. — Two vectors *equal, parallel,* and having the *same direction* are termed *equipollent.* This geometric equivalence is expressed by writing a dash over the two vectors, as

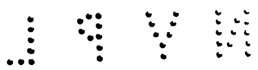
$$\overline{P_1} = \overline{P_2} \qquad [2]$$

which asserts that P_1 is equipollent to P_2. Two vectors which are equal and parallel but have opposite directions are termed *equal and opposite,* and this is expressed by the equipollence

$$\overline{P_1} = -\overline{P_2} \qquad [3]$$

If they have the same origin, or are situated on the same straight line, they are termed *equal and directly opposite.* Figure 2 illustrates the different cases of *equal and directly opposite vectors.* This is again expressed by the equipollence

$$\overline{P_1} = -\overline{P_2} \qquad [4]$$



which, it will be observed, takes no account of the origins of the vectors.

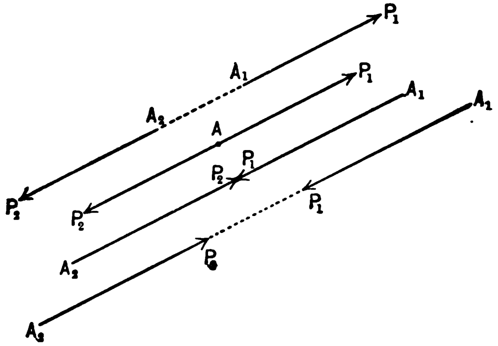


Fig. 2.

3. Geometric sum, or composition of vectors.—Consider (fig. 3_1) a system of vectors

$$P_1, P_2, \ldots P_{n-1}, P_n$$

distributed in any manner in space. Displace them parallel to themselves so as to place them end to end in *any* order—in the order of their indices, for instance. To fix the ideas suppose n equal to six. By this means there is formed an *open* polygonal contour (fig. 3_2), plane or twisted, of which the sides

$$1, 2, 3, 4, 5, 6$$

are equipollent to the given vectors. This polygon is termed the *polygon of the given vectors*, and the vector AP_6, which *closes* it, is, *by definition*, the *geometric sum* of the given vectors. Call this sum S. Then this definition is expressed *analytically* by the equipollence

$$\overline{S}=\overline{P_1}+\overline{P_2}+\overline{P_3}+\overline{P_4}+\overline{P_5}+\overline{P_6}$$

and in the general case by

$$\overline{S} = \sum_{i=1}^{i=n} \overline{P_i} \qquad [5]$$

Remark.—This extension of the word *sum*, extremely convenient in mechanics, is justified by the fact that these geometric sums possess the same properties as arithmetic

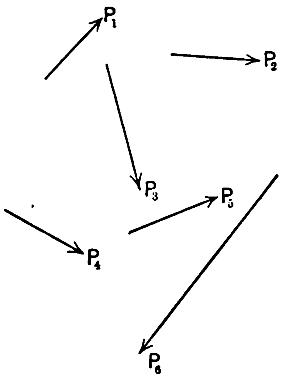


Fig. 3_1.

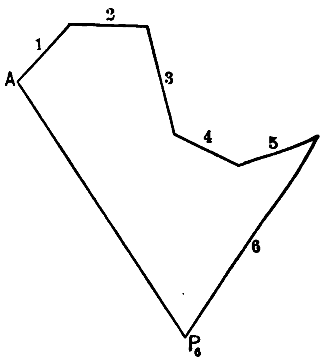


Fig. 3_2.

and algebraic sums, which they include as particular cases. This results from the following *geometric* theorems which are easily proven, but which I shall not stop to verify.

Theorem I.—*The geometric sum of a system of vectors is independent of the order of summation.*

Theorem II.—*The geometric sum of a system of vectors is not changed by replacing certain of them by their partial sums, or conversely by replacing certain of them by others of which they are the partial sums.*

Theorem III$_1$.—*If all the vectors of a system be increased or decreased in the same ratio, their geometric sum is increased or decreased in this ratio.*

THEOREM III_2.—*To multiply the geometric sum by any number, it is sufficient to multiply each vector of the system by this number.*

4. **Analytic expression of geometric sum.**—1° *Two vectors.*—Forming the polygon (fig. 4) of the two vectors, P_1 and P_2, this reduces to a triangle. Let α be the angle between the *positive* directions of P_1 and P_2. Their geometric sum is, by definition, the third side of this triangle, and

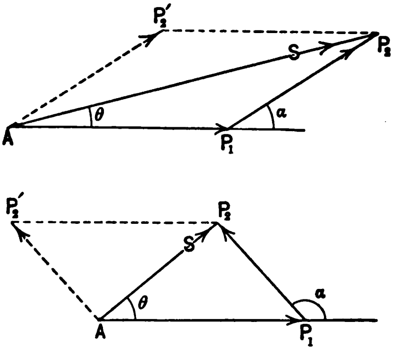


FIG. 4.

$$\overline{S}=\overline{P_1}+\overline{P_2}$$

The *magnitude* of S is given by the *algebraic* equation

$$S^2=P_1{}^2+P_2{}^2+2P_1P_2\cos\alpha$$

To determine θ, the angle between the positive directions of S and P_1, we have the relation

$$\frac{P_2}{\sin\theta}=\frac{S}{\sin\alpha}$$

Whence
$$\sin\theta = \frac{P_2}{S}\sin\alpha \qquad [6]$$

Also
$$P_2^2 = P_1^2 + S^2 - 2P_1S\cos\theta$$

Whence
$$\cos\theta = \frac{S^2 + P_1^2 - P_2^2}{2P_1S}$$

or
$$\cos\theta = \frac{P_1 + P_2\cos\alpha}{S} \qquad [7]$$

Since $0^\circ \overset{=}{<} \theta \overset{=}{<} 180^\circ$, [7] determines which quadrant it lies in.

REMARK.—From the figure it is evident that if we complete the parallelogram having P_1 and P_2 as sides, the vector P_2' is equipollent to P_2, and S is the *diagonal* of the parallelogram having P_1 and P_2' for sides, whence the

THEOREM.—*If from the same point two vectors be drawn equipollent to the given vectors, the* DIAGONAL *of the parallelogram constructed on these two vectors as sides is equipollent to the geometric sum of the given vectors.*

COROLLARY.—If $\alpha = 90^\circ$, then

$$S^2 = P_1^2 + P_2^2$$

and
$$\cos\theta = \frac{P_1}{S}$$

or
$$\tan\theta = \frac{P_2}{P_1}$$

2° *Three vectors.*—The geometric sum S (fig. 5) of three vectors P_1, P_2, and P_3 is, by definition, the vector AS. Denoting by α, β, γ the angles between the positive directions of P_1 and P_2, P_2 and P_3, P_3 and P_1, the magnitude of S is given by

$$S^2 = P_1^2 + P_2^2 + P_3^2 + 2P_1P_2 \cos \alpha \\ + 2P_2P_3 \cos \beta + 2P_3P_1 \cos \gamma \quad [8]$$

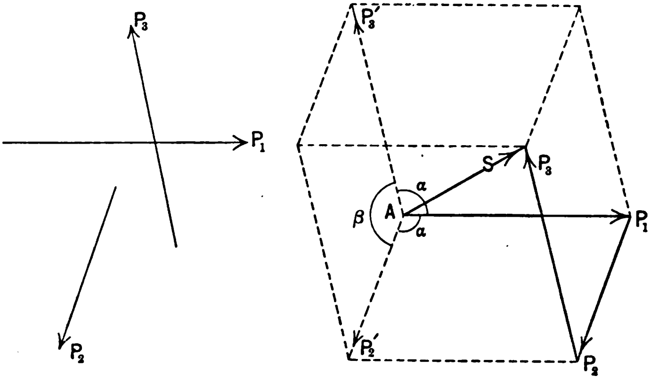


FIG. 5.

The angles θ_1, θ_2, θ_3 made by the positive direction of S and the positive directions of P_1, P_2, and P_3 are determined by the relations

$$\left.\begin{array}{l} P_1^2 + S_2 - 2P_1S \cos \theta_1 = P_2^2 + P_3^2 + 2P_2P_3 \cos \beta \\ P_2^2 + S_2 - 2P_2S \cos \theta_2 = P_3^2 + P_1^2 + 2P_3P_1 \cos \gamma \\ P_3^2 + S_2 - 2P_3S \cos \theta_3 = P_1^2 + P_2^2 + 2P_1P_2 \cos \alpha \end{array}\right\} \quad [9]$$

REMARK.—It is again evident from the figure that if at at A vectors P_2' and P_3' be drawn equipollent to P_2 and P_3, the vector AS is the diagonal of the parallelopiped constructed on AP_1, AP_2, AP_3 as edges, whence the

THEOREM.—*If from the same point vectors be drawn equipollent to three given vectors, the diagonal of the parallelopiped constructed on these as edges is equipollent to the geometric sum of the given vectors.*

5. Geometric difference.—The *geometric difference* of two vectors P_1 and P_2 is, *by definition,* the vector which added to one of them gives the other.

Subtracting P_1 *from* P_2 and calling the difference Q, we have the equipollence

$$\overline{P_2}=\overline{P_1}+\overline{Q}$$

From Art. 4 it follows that P_2 is equipollent to the diagonal of the parallelogram having P_1 and Q as sides. If then from A (fig. 6), the origin of P_1 and P_2, a vector $-P_1$,

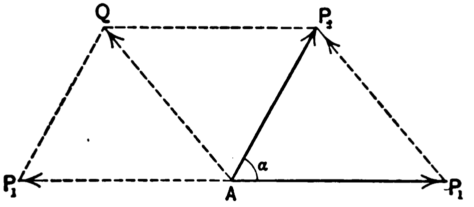


Fig. 6.

equal and opposite to P_1, be drawn, Q is equipollent to the diagonal of the parallelogram having $-P_1$ and P_2 for sides. Q is then the geometric sum of P_2 and $-P_1$, which is expressed by the equipollence

$$\overline{Q}=\overline{P_2}+(-\overline{P_1})$$

or

$$\overline{Q}=\overline{P_2}-\overline{P_1} \qquad [10]$$

which may be looked upon as a consequence of the equipollence

$$\overline{P_2}=\overline{P_1}+\overline{Q}$$

It follows, then, that an equipollence involving three vectors may be treated as an algebraic equation so far as the trans-

position of its terms is concerned, and by similar reasoning an equipollence involving any number of vectors may be so treated.

It is evident from the figure that the vector P_1P_2 is also equipollent to Q, and we have then the following geometrical construction for the difference of two vectors.

From a point A draw two vectors P_1 *and* P_2 *equipollent to the given vectors and complete the triangle by connecting their extremities. The third side of this triangle is equipollent to the difference of the two vectors. The direction of this difference is from* P_1 *to* P_2 *if* P_1 *is subtracted from* P_2*, and from* P_2 *to* P_1 *if* P_2 *is subtracted from* P_1.

6. Analytic expression of geometric difference.—If α be the angle between P_1 and P_2, then

$$Q^2 = P_1{}^2 + P_2{}^2 - 2P_1P_2 \cos \alpha \qquad [11]$$

which determines the arithmetic value of Q.

Replacing P_1 by $-P_1$ and S by Q in [7], we have

$$\cos \theta = \frac{-P_1 + P_2 \cos \alpha}{Q} \qquad [12]$$

which determines the angle θ between P_1 and Q.

7. Decomposition of a vector.—To decompose a vector S is, *by definition*, to find a system of vectors

$$P_1 \ldots P_n$$

of which S is the geometric sum.

A system of two vectors P_1 and P_2 of which S is the geometric sum is easily found. In fact, it is sufficient to construct any parallelogram of which S is the diagonal.

The sides P_1 and P_2 of this parallelogram are then equipollent to the required vectors. Similarly to decompose a vector S into a system of three vectors it is sufficient to construct a parallelopiped of which S is the diagonal. The three edges of this parallelopiped issuing from the vertex which is the origin of S are then equipollent to the required vectors. Since an infinite number of such parallelograms or parallelopipeds can be constructed, the decomposition can be effected in an infinite number of ways, and to obtain a *unique* decomposition *one* vector must be assigned in the parallelogram and *two* in the parallelopiped.

Instead of assigning one or two of the vectors, two or three of the directions may be assigned, and the decomposition is unique since there can be but one parallelogram or parallelopiped with assigned directions for its edges.

8. Analytic expression of the components.—1° If one of the component vectors, P_1 say, is assigned in magnitude and direction, the magnitude and direction of P_2 (fig. 7) are given by

$$\left.\begin{aligned} P_2^2 &= P_1^2 + S^2 - 2P_1S\cos\theta_1 \\ \sin\theta_2 &= \sin\theta_1\frac{P_1}{S} \end{aligned}\right\} \quad [13]$$

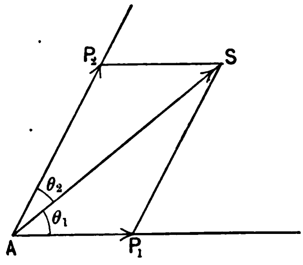


FIG. 7.

2° If the two directions θ_1 and θ_2 are assigned, then the magnitudes are given by the equations

$$\left.\begin{aligned} P_1 &= S\frac{\sin\theta_2}{\sin(\theta_1+\theta_2)} \\ P_2 &= S\frac{\sin\theta_1}{\sin(\theta_1+\theta_2)} \end{aligned}\right\} \quad [14]$$

REMARK. — The analytic expressions for the directions and magnitudes when S is decomposed into three components P_1, P_2, and P_3 are easily obtained from the geometry of the parallelopiped.

If $\theta_1+\theta_2=90°$, or the assigned directions are orthogonal, then

$$P_1=S\cos\theta_1$$
$$P_2=S\cos\theta_2$$

9. Projection of a vector on a line.—The *projection* of a vector AP on a line OX (fig. 8) is the vector A_xP_x obtained by projecting the line AP on OX. The projection of A is, by definition, the origin of A_xP_x.

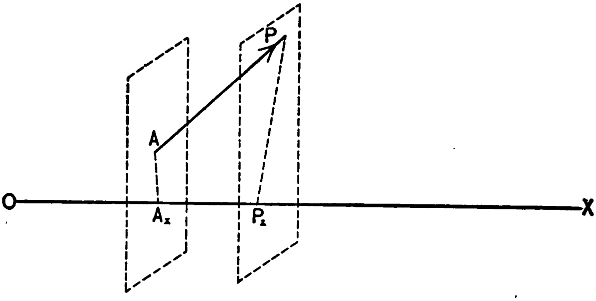


FIG. 8.

Denoting by θ the angle between P and OX, we have

$$P_x=P\cos\theta \qquad [15]$$

THEOREM I.—*The algebraic sum of the projections of two vectors on a line is equal to the projection of the geometric sum of the two vectors on that line.*

Let (fig. 9) P_1 and P_2 be the given vectors and S their geometric sum. Then $A_xP_{x_1}$, $A_xP_{x_2}$, and A_xS_x are their projections on OX,

and $$A_xS_x = A_xP_{x_1} + P_{x_1}S_x$$

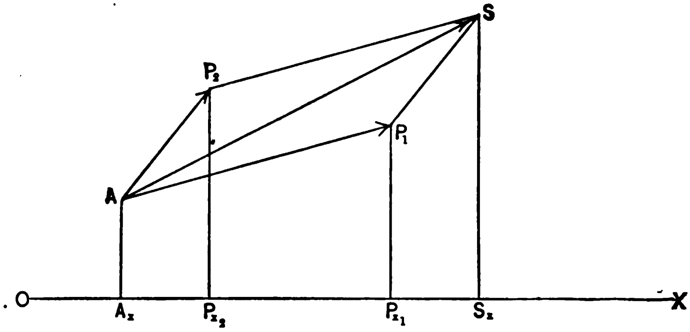


FIG. 9.

But P_1S is equipollent to AP_2, and therefore

$$P_{x_1}S_x = A_xP_{x_2}$$

Hence $$A_xS_x = A_xP_{x_1} + A_xP_{x_2}$$

or denoting by θ, θ_1, and θ_2 the angles between S, P_1, P_2, and OX

$$S \cos \theta = P_1 \cos \theta_1 + P_2 \cos \theta_2 \qquad [16]$$

This same reasoning can be extended to a system of n vectors by first taking two of them, then their geometric sum and a third and so on until the last vector is reached, whence

THEOREM II.—*The algebraic sum of the projections of a system of n vectors on a line equals the projection of the geometric sum of the system on that line, or*

$$S \cos \theta = P_1 \cos \theta_1 + \ldots + P_n \cos \theta_n$$

or

$$S \cos \theta = \sum_1^n P_i \cos \theta_i \qquad [17]$$

10. Analytic expression of geometric sum referred to axes.—The two theorems of the last section give a simple method of determining the geometric sum of a system of vectors. Choosing the three perpendicular axes OX, OY, and OZ and projecting all the vectors of the system on these axes, the sum of the projections on each axis gives the projection of the geometric sum on that axis, which is the edge of the rectangular parallelopiped of which the geometric sum is the diagonal and since when the three edges are determined the diagonal is determined, the geometric sum is determined in magnitude and direction.

Writing

$$\left.\begin{aligned} S_x &= \Sigma P_x \\ S_y &= \Sigma P_y \\ S_z &= \Sigma P_z \end{aligned}\right\}$$

then

$$S^2 = S_x{}^2 + S_y{}^2 + S_z{}^2 \qquad [18]$$

and calling θ_x, θ_y, and θ_z the angles between S and OX, OY, OZ

$$\left.\begin{aligned}\cos\theta_x&=\frac{S_x}{S}\\ \cos\theta_y&=\frac{S_y}{S}\\ \cos\theta_z&=\frac{S_z}{S}\end{aligned}\right\}\qquad[19]$$

Corollary.—If the vectors all lie in one plane, then referring them to two rectangular axes OX and OY in that plane

$$S^2=S_x^2+S_y^2$$

$$\cos\theta_x=\frac{S_x}{S}$$

$$\cos\theta_y=\frac{S_y}{S}$$

11. Geometric derivatives. — 1° *Vector with a fixed origin.*—Consider AP (fig. 10) whose origin A is fixed and whose extremity P describes a curve C.

Let AP be its position at the instant t and AP_1 its position at the instant $t+\Delta t$.

The *geometric increment* of P during the time Δt is the geometric difference

$$\overline{PP_1}=\overline{P_1}-\overline{P}\equiv\overline{\Delta P}$$

The vector

$$\overline{PE}=\frac{\overline{\Delta P}}{\Delta t}$$

tends to the limiting vector PP', tangent to C at P, as Δt tends to zero, so that

$$\overline{PP'} = \lim \left(\frac{\overline{\Delta P}}{\Delta t}\right)_{\Delta t = 0}$$

and this vector PP' is called the *geometric derivative* of P at the instant t.

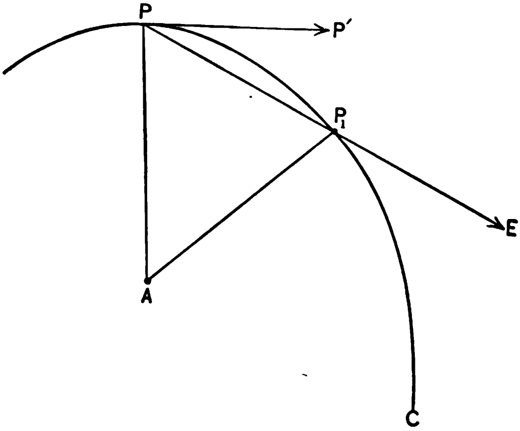


FIG. 10.

I shall use the symbol

$$\frac{D}{Dt}$$

to denote a geometric derivative in distinction from the symbol

$$\frac{d}{dt}$$

used for algebraic derivatives.

We may then write

$$\frac{\overline{DP}}{Dt}=\lim\left(\frac{\overline{\varDelta P}}{\varDelta t}\right)_{\varDelta t=0} \qquad [20]$$

2° *Vector with moving origin.*—To obtain the geometric derivative of the vector PP' with moving origin P (fig. 11) at the instant t we construct at the *fixed* point P a vector equipollent to the vector PP' and take its geometric derivative, and this is defined as the second geometric derivative of AP, so that

$$\overline{PP''}=\frac{\overline{DP'}}{Dt}=\frac{\overline{D^2P}}{Dt^2} \qquad [21]$$

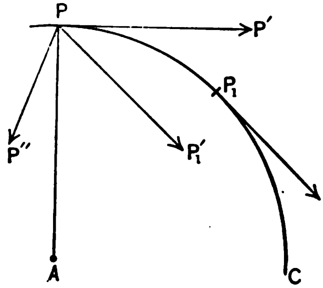


FIG. 11.

where PP'' represents a vector equipollent to the geometric derivative of PP'.

12. Projection of the geometric derivative of a vector on a plane and on an axis. — Let the moving vector AP (fig. 12) be projected on the fixed plane Π, and denote projections on this plane by the symbol $(\ )_\pi$

Then $$(\overline{PP_1})_\pi=\overline{pp_1}$$

or $$(\overline{\varDelta P})_\pi=\overline{\varDelta p}$$

Hence $$\left(\frac{\overline{\varDelta P}}{\varDelta t}\right)_\pi=\frac{\overline{\varDelta p}}{\varDelta t}$$

since ΔP and $\frac{\Delta P}{\Delta t}$ are vectors making the same angle with Π.

$$\text{Therefore } \lim \left(\frac{\overline{\Delta P}}{\Delta t}\right)_{\pi}\Bigg]_{\Delta t=0} = \lim \frac{\overline{\Delta p}}{\Delta t}\Bigg]_{\Delta t=0}$$

or
$$\left(\frac{\overline{DP}}{Dt}\right)_{\pi} = \frac{\overline{Dp}}{Dt} \qquad [22]$$

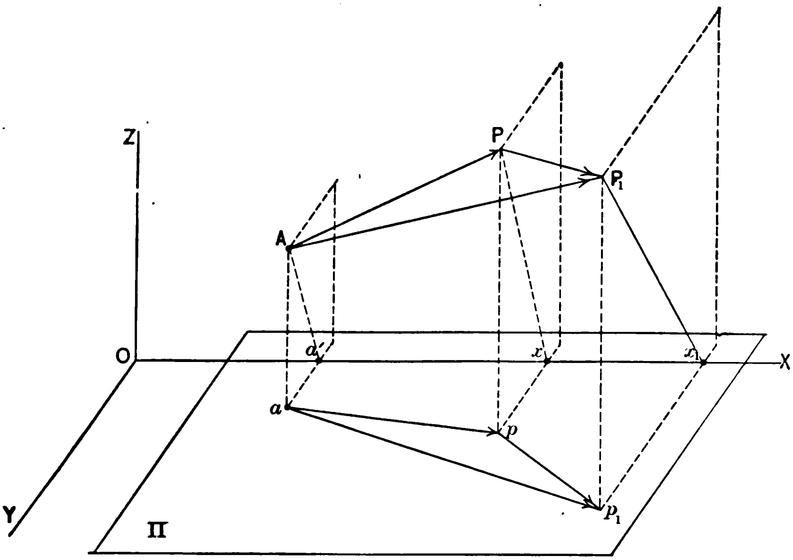


FIG. 12.

HENCE, *the projection on any plane of the geometric derivative of a vector is equipollent to the geometric derivative of the projection of the vector on the plane.*

Let OX be any axis—not necessarily connected with the plane Π, but for convenience taken in that plane. Denote projections on OX by the symbol $(\ )_x$.

Then $$(PP_1)_x = x_1 - x$$

or $$(\Delta P)_x = \Delta x$$

Hence $$\left(\frac{\Delta P}{\Delta t}\right)_x = \frac{\Delta x}{\Delta t}$$

and $$\lim \left(\frac{\Delta P}{\Delta t}\right)_x \Big]_{\Delta t = 0} = \lim \frac{\Delta x}{\Delta t}\Big]_{\Delta t = 0}$$

or $$\left(\frac{DP}{Dt}\right)_x = \frac{dx}{dt} = \frac{d(x-a')}{dt}. \qquad [23]$$

Hence, *the projection of the geometric derivative of a vector on an axis equals the algebraic derivative of the projection of the vector on that axis.*

If θ denote the angle that AP makes with OX then

$$x - a' = r \cos \theta$$

and $$\frac{d(x-a')}{dt} = \frac{dr}{dt} \cos \theta - r \sin \theta \frac{d\theta}{dt}$$

where r is the length of AP.

Hence $$\left(\frac{DP}{Dt}\right)_x = \frac{dr}{dt} \cos \theta - r \sin \theta \frac{d\theta}{dt} \qquad [24]$$

Corollary. — If AP be perpendicular to OX, then $\theta = 90°$ and

$$\left(\frac{DP}{Dt}\right)_x = -r\left(\frac{d\theta}{dt}\right)_{\theta=90°} \qquad [25]$$

13. Projection of the geometric derivative of a vector on the vector itself.—Making $\theta = 0$ in [24] we have for the projection of the geometric derivative of a vector on the vector the expression

$$\left(\frac{DP}{Dt}\right)_p = \frac{dr}{dt} \qquad [26]$$

14. Applications.—*First problem: What is the necessary and sufficient condition that the geometric derivative of a vector be constantly directed along the vector?*

In this case the geometric derivative equals its projection on the vector itself,

whence
$$\frac{DP}{Dt} = \left(\frac{DP}{Dt}\right)_p = \frac{dr}{dt}$$

Hence, projecting on the X-axis,

$$\left(\frac{DP}{Dt}\right)_x = \frac{dr}{dt}\cos\theta$$

and therefore

$$\frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt} = \frac{dr}{dt}\cos\theta$$

whence $$r \sin \theta \frac{d\theta}{dt} = 0$$

or $$\theta = \theta_0, \text{ a const.}$$

THEREFORE, *the necessary and sufficient condition that the geometric derivative of a vector be constantly directed along the vector is that the orientation of the vector be constant.*

Second problem: What is the necessary and sufficient condition that the geometric derivative of a vector be constantly normal to the vector?

In this case the projection of the geometric derivative on the vector is zero,

whence $$\frac{dr}{dt} = 0$$

and $$r = r_0, \text{ a const.}$$

THEREFORE, *the necessary and sufficient condition that the geometric derivative of a vector be constantly normal to the vector is that the length of the vector be constant.*

Third problem: What is the necessary and sufficient condition that the geometric derivative of a vector be zero?

If the geometric derivative be zero, then its projection on any axis is zero,

whence $$\frac{dx}{dt} = 0$$

or $$x = x_0, \text{ a const.}$$

similarly $y = y_0$, a const.

$z = z_0$, a const.

and hence the extremity P is fixed.

THEREFORE, *the necessary and sufficient condition that the geometric derivative of a vector be constantly zero is that the vector be constant both in orientation and magnitude.*

CHAPTER II

KINEMATICS OF A POINT. GENERAL THEOREMS

15. Definitions. Rest and motion.—16. Time.—17. Motion of a point. Path.—18. Equations of motion.—19. Rectilinear motion.—20. Units of time and of length.—21. Case where it is necessary to specify the units.—22. Homogeneous equations.—23. Change of units.—24. Uniform rectilinear motion. Velocity.—25. Equation of uniform motion.—26. Accelerated rectilinear motion.—27. Numerical examples.—28. Knots and nautical miles.—29. Equation of accelerated motion in terms of the mean velocity.—30. Acceleration.—31. Equation of uniformly accelerated motion.—32. Discussion of the motion by means of the equation of motion.—33. Case where the displacement is taken as independent variable.

15. Definitions. Rest and motion.—When a point is spoken of as at rest or in motion it is always understood that this rest or motion takes place with respect to other points in its neighborhood. A point immovable on the Earth's surface is at rest with respect to all other fixed points on the Earth, but is moving with respect to points on the Sun, since the Earth is itself moving relative to the Sun. Hence, any *relative* motion is observable, but it is convenient in kinematics to choose some system of points to which the motion of all other points may be referred to regard this system as *absolutely fixed,* by *definition.* Motion with respect to this system is then termed absolute motion. Thus, for points on the Earth, a system of points immovable on the Earth's surface may be chosen as the *absolutely fixed system*, and the motion of all other points referred to them. To discuss the motion of the Earth and planets a second system of *absolutely fixed points* must be chosen. These may be conveniently taken on the Sun, while to discuss the motion of the solar system the so-called *fixed stars* must be chosen as points of reference. There being no system of points to which the motion of these stars can be referred, discussion cannot go beyond them.

But while, in kinematics, the choice of the *absolutely fixed system* is perfectly *arbitrary*, it is *no longer so in mechanics*, and there we shall see that the fixed stars *must* be chosen as the system of reference.

16. Time.—To define the time when the moving point is at a given position, it is referred to a certain instant, called the *initial instant*, and the number of *mean solar seconds* before or after the initial instant is given. Hence a time t designates an instant t seconds *before* or t seconds *after* the initial instant according as t is *positive* or *negative*, that is, according as it has the plus or minus sign prefixed.

17. Motion of a point. Path.—A point is said to move with respect to some arbitrarily fixed system of points when *at least one* of the distances between it and the points of the fixed system changes. The curve described by the moving point is called its *path* or *trajectory*.

18. Equations of motion.—Referring the point to a system of rectangular axes $O—XYZ$, its coordinates will vary with the time and are therefore functions of t. Suppose them to be given by the following equations:

$$x=\phi(t) \qquad y=\chi(t) \qquad z=\psi(t) \qquad [27]$$

When these functions are known, the motion is completely known, for the path may be obtained by eliminating

t between these three equations taken in pairs, which will give two surfaces whose intersection is the path, or equations [27] may be regarded as the parametric equations of the path. The position of the point on the path is also known, for as soon as t is given the coordinates of the point at that instant are determined by [27]. These equations [27] are termed the *equations of motion,* and the problem of finding the motion of a point is the problem of finding these equations.

A second method of fixing the position of the point M in its path is the following: Its distance s from some fixed point M_0 on the path, *measured along the path,* may be given at each instant, the direction of motion being *chosen* as positive. The equation of motion then has the form

$$s=f(t) \qquad [28]$$

the path being supposed given.

19. Rectilinear motion.—If the path is a straight line, the motion is termed *rectilinear*. Choosing the path as the axis of X, the two methods of fixing the position of the point M become identical, and the equation of motion has the form

$$x=f(t) \qquad [29]$$

the point M_0 being the origin O.

Now for a given value of t, x can have one value only and hence the function $f(t)$ is *uniform,* and if a *continuous* sequence of values of t be substituted in [29] we must get a *continuous* sequence of values for x, since M cannot pass from one point on its path to another without passing through the intermediate points. This same equation [29] permits us to solve the inverse problem, that is, to answer the question: What are the times when M passes a given point on its path? In this case x is given and we require t. It is necessary, therefore, to solve [29] for t, which we may suppose done, with the result that

$$t=\phi(x) \tag{30}$$

which is the equation of motion in its *second* form. Often the conditions of the problem lead us to an equation of this form, which must be solved for x to get the *first* form. For theoretical considerations, however, the *first* form is much to be preferred. As has already been remarked, $f(t)$ can have *only one* value for a given value of t, but $\phi(x)$ may have more than one value for a given value of x, since the moving point may pass the same place in its path any number of times.

20. Units of time and of length.—As has already been remarked in Art. 16, time is expressed in mean solar seconds. The unit may be taken as the second itself or any multiple thereof, as the minute, hour, day, or year.

The unit of length is usually the metre or the kilometre. Unless the contrary is stated the second and metre will be used as the units.

21. Case where it is necessary to specify the units.—In general a numerical equation of motion

$$x=f(t)$$

arbitrarily given has no meaning unless the units are stated. Thus the equation

$$x=t^2$$

gives $x=3600$ metres in 1 minute if the second and metre are adopted, but gives $x=1$ kilometre in 1 minute if the kilometre and minute are adopted.

22. Homogeneous equations. — The equations which, instead of being given empirically, express laws of nature cannot depend on the choice of units and must therefore be homogeneous, that is, are necessarily of the form

$$\frac{x}{\lambda}=f\left(\frac{t}{\tau}\right), \qquad [31]$$

where λ is a length and τ a time expressed in the *same* units as x and t, and hence if the units be changed the equation is not affected since the ratios $\frac{x}{\lambda}$ and $\frac{t}{\tau}$ remain the same.

23. Change of units.—If we have on the other hand a non-homogeneous equation, such as would result from numerical observations,

$$x=f(t)$$

with determined units of time and length, it is easy to see what this becomes when other units of time and length are used. Suppose the new unit of length to be $\left(\frac{1}{\lambda}\right)$ of the old, and the new unit of time $\left(\frac{1}{\tau}\right)$ of the old. Then it is clear that a determined length which contains the old unit x times contains the new one λx times and its new *numerical* value will be

$$x_1=\lambda x, \qquad \text{whence} \qquad x=\frac{x_1}{\lambda}$$

Similarly $t_1 = \tau t$, whence $t = \frac{t_1}{\tau}$

and the equation

$$x = f(t)$$

becomes

$$\frac{x_1}{\lambda} = f\left(\frac{t_1}{\tau}\right)$$

Consider again, as an example, the equation

$$x = t^2$$

and suppose the metre and second to be the units. It may then be written

$$\frac{x}{1^{\mathrm{m}}} = \left(\frac{t}{1^{\mathrm{s}}}\right)^2$$

so that at the end of a minute

$$x = 3600^{\mathrm{m}}$$

Taking the kilometre and minute as new units,

$$\frac{1}{\lambda} = 1000 \quad \text{and} \quad \frac{1}{\tau} = 60$$

so that the equation becomes

$$1000x_1 = (60t_1)^2 = 3600t_1^2$$

or

$$x_1 = 3.6t_1^2$$

For $t_1 = 1$ we have $x_1 = 3.6$, which means that at the end of 1 minute the abscissa of the moving point is 3.6^{km}, which is what we found with the old units. The equations

$$\frac{x}{1^{\text{m}}} = \left(\frac{t}{1^{\text{s}}}\right)^2$$

$$\frac{x_1}{1^{\text{km}}} = \left(\frac{t_1}{1^{\text{min}}}\right)^2$$

are then equivalent.

24. Uniform rectilinear motion. Velocity. — Motion is termed uniform rectilinear when the path is a straight line and equal distances are passed over in equal times. Choosing the X-axis as the path and counting time from the instant when the point is at the origin, O, its position at any other instant is given by its abscissa. Let M be its position at the instant t (fig. 13) and M_1 its position at the instant $t + \Delta t$, Δt being positive. The geometric magnitude MM_1 has the algebraic value Δx, where this represents the increase in the abscissa x of the moving point in time Δt. If in the direction MM_1 a vector MV of length

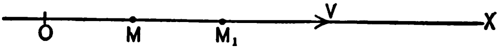


FIG. 13.

$\frac{\Delta x}{\Delta t}$ be drawn, this vector is, *by definition*, the *velocity of the uniform motion* at the instant t. The length of the vector MV is termed the *algebraic value* of the velocity, or the *speed*, and is denoted by v. It is positive or negative according as MV is directed in the positive or negative direction along OX. In magnitude and sign, then,

$$v=\frac{\Delta x}{\Delta t}$$

Since the motion is uniform, that is, since equal distances are passed over in equal times,

$$\Delta x=k\Delta t$$

no matter what values t and Δt have, and

$$v=\frac{\Delta x}{\Delta t}=k, \text{ a constant.}$$

HENCE, *the velocity in uniform rectilinear motion is constant in magnitude and direction.*

REMARK.—If $\Delta t=1$, then $v=\Delta x$, whence *the velocity in uniform rectilinear motion is in magnitude and direction the space travelled in unit of time.*

25. Equation of uniform motion.—Since v is the distance travelled in unit time, the distance travelled in t units is vt, and if x_0 be the position of the point at instant $t=0$, its position at the instant t is given by

$$x=x_0+vt \qquad [32]$$

26. Accelerated rectilinear motion.—Every motion not uniform is *accelerated,* by *definition.* Let the accelerated motion be defined by $x=f(t)$. The displacement MM_1 which the point undergoes when t increases by Δt is a vector whose algebraic value is Δx.

Velocity.—If in the direction MM_1 (fig. 14) a vector MW of length $\frac{\Delta x}{\Delta t}$ be drawn, this vector, whose algebraic value is $\frac{\Delta x}{\Delta t}$, is termed the *mean velocity* of the moving point during the interval Δt. As Δt tends to zero, this vector MW tends to a limiting vector MV, which is termed the *velocity* at the *instant t.* The algebraic value of MV, termed the *speed* at instant t, is

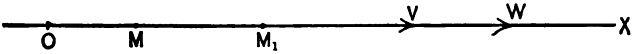


FIG. 14.

$$v=\lim \frac{\Delta x}{\Delta t}\Big]_{\Delta t=0} = \frac{dx}{dt}$$

From the equation of motion

$$x=f(t)$$

we have by differentiation

$$v=\frac{dx}{dt}=f'(t) \qquad [33]$$

27. Numerical examples.—A mean speed is thus seen to be the ratio of two magnitudes essentially positive and of different nature; the ratio of a length to a time. These two magnitudes can be expressed in two absolutely independent units, and it follows, then, that the numerical value of a mean velocity does not have any meaning unless the units are specified, for the same velocity can be expressed by different numbers depending on the units chosen. Suppose, for example, a train takes five hours to go from Washington to New York. The distance is 354 km. (about), and hence the mean speed between Washington and New York is

$$v_m = \frac{354}{5} = 70.8 \text{ km. per hour}$$

$$= 70.8 \frac{\text{km.}}{\text{hour}}$$

Expressing the distance in miles,

$$v_m = \frac{220}{5} = 44 \text{ miles per hour}$$

$$= 44 \frac{\text{miles}}{\text{hour}}$$

28. Knots and nautical miles.—The speed of a ship is usually expressed in knots. This is an abbreviation of language, and the units here understood are the nautical mile and the hour. To say then that a ship has a speed of 20 knots is to say that it has a speed of 20 nautical miles per hour, or

$$v = 20 \frac{\text{naut. miles}}{\text{hour}}$$

A nautical mile is the length of 1′ of the earth's meridian or about 1852 metres. A speed of one knot is equivalent to a speed of .514 metres per second, or

$$1\frac{\text{naut. mile}}{\text{hour}}=.574\frac{\text{metres}}{\text{second}}$$

29. Equation of accelerated motion in terms of the mean velocity.—If the mean velocity during any interval is known, the distance travelled is at once obtained from the relation

$$v_m=\frac{\Delta x}{\Delta t}=\frac{x_1-x}{t_1-t}$$

which gives

$$x_1=x+v_m(t_1-t)$$

so that knowing the position of the point at the instant t and its mean velocity during the interval t_1-t we have its position at the instant t_1.

30. Acceleration.—Uniform rectilinear motion is characterized by the property that at each instant the velocity is the same in magnitude and direction. In accelerated motion the velocity varies from instant to instant, and it is useful to take account in an exact manner of this variation. The notion of the vector *acceleration* is thus obtained. Let the moving point be at M (fig. 15) at the instant t, and have there a velocity MV. At instant t_1 let M_1 and M_1V_1 denote its position and velocity, and let $v_1>v$. In the

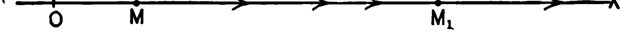


Fig. 15.

direction MM_1 construct the vector MK of length

$$\frac{v_1-v}{t_1-t}=\frac{\Delta v}{\Delta t}$$

This vector MK is termed the mean acceleration during the interval t_1-t. Let Δt tend to zero, then MK tends to a limiting vector MJ, which is termed the acceleration at the instant t.

The algebraic value or magnitude of MJ is

$$j=\lim \frac{v_1-v}{t_1-t}\Big]_{t_1=t}=\lim \frac{\Delta v}{\Delta t}\Big]_{\Delta t=0}$$

or
$$j=\frac{dv}{dt} \qquad [34]$$

But since
$$v=\frac{dx}{dt}$$

we have
$$j=\frac{d^2x}{dt^2} \qquad [35]$$

If the acceleration J is the same for all points of the path, the motion is termed *uniformly accelerated,* and the change in the velocity is proportional to the time. If the velocity decreases with the time the acceleration is negative and J is drawn in the opposite direction

31. Equation of uniformly accelerated motion.—Since in this case J is constant we have

$$\frac{d^2x}{dt^2}=\gamma, \text{ a constant}$$

whence
$$\frac{dx}{dt}=\gamma t+c$$

and
$$x=\tfrac{1}{2}\gamma t^2+ct+c_1$$

But making $t=0$ we have

$$\left(\frac{dx}{dt}\right)_0=c$$

and
$$x_0=c_1$$

whence
$$x=x_0+v_0t+\tfrac{1}{2}\gamma t^2 \qquad [36]$$

Where x_0 and v_0 are the displacement and velocity at the instant $t=0$.

32. **Discussion of the motion by means of the equation of motion.**—If we are given the equation of motion

$$x=f(t)$$

the position, velocity, and acceleration at any instant may be obtained by differentiating this equation with respect to t, for

$$v=\frac{dx}{dt}=f'(t)$$

$$j=\frac{d^2x}{dt^2}=f''(t)$$

Conversely, if we are given the acceleration at every instant, that is, if we are given the acceleration *as a func-*

tion of t, and the velocity and displacement at *one instant,* we can determine the equation of motion. In order to do this it is necessary to integrate the differential equation of the second order

$$\frac{d^2x}{dt^2}=j(t)$$

whose solution will contain *two* constants of integration, such that

$$x=f(t,\ c,\ c_1)$$

But c and c_1 may be determined by substituting in

$$x=f(t,\ c,\ c_1)$$

and
$$v=\frac{dx}{dt}=f'(t,\ c,\ c_1)$$

the known values of x and v for the *given* value of t.

33. **Case where the displacement is taken as independent variable.**—It is often useful, instead of determining the circumstances of the motion at *each instant,* to determine them at *each point* of the path, that is, to express them in terms of the displacement instead of the time. This is equivalent to taking x instead of t as the independent variable, and expressing the equation of motion in the form

$$t=\phi(x)$$

This may be given directly or it may be obtained by solving the equation

$$x=f(t)$$

for t.

To obtain the velocity and acceleration in terms of x, we have

$$v=\frac{dx}{dt}=\frac{1}{\frac{dt}{dx}}=\frac{1}{\phi'(x)} \tag{37}$$

$$j=\frac{d^2x}{dt^2}=-\frac{\frac{d^2t}{dx^2}}{\left(\frac{dt}{dx}\right)^3}=-\frac{\phi''(x)}{\phi'^3x} \tag{38}$$

CHAPTER III

APPLICATIONS TO ORDINARY RECTILINEAR MOTIONS

A. UNIFORM MOTION

34. Properties of uniform motion.—35. Problem.

B. UNIFORMLY ACCELERATED MOTION

36. Properties of uniformly accelerated motion. — 37. Remark.— 38. Equations obtained by taking the space described as independent variable.

C. PERIODIC MOTIONS

39. Equations of periodic motion.—40. Equation of harmonic motion. —41. Amplitude, elongation, frequency, argument, phase.— 42. Velocity and acceleration.—43. Discussion of the motion.— 44. Influence of the phase.

A. UNIFORM MOTION

34. Properties of uniform motion.—The equation of uniform motion was deduced in the following form:

$$x = x_0 + vt$$

From this we have the following

THEOREM I.—*In a uniform motion the abscissa of the moving point is a linear function of the time.*

Conversely, if the motion is uniform, then the velocity is constant, or

$$v = \frac{dx}{dt} = c, \text{ a constant}$$

Now any equation of motion of the form

$$x = a + bt$$

gives

$$v = \frac{dx}{dt} = b$$

from which follows the

THEOREM II.—*Any rectilinear motion whose displacement is a linear function of the time is uniform.*

35. **Problem.**—*Given the instants of passage of the uniformly moving point at two points on its path, find the equation of motion.*

Let x_0 and x_1 be the abscissas of the points M_0 and M_1 which the moving point passes at instants t_0 and t_1.

The velocity is then

$$v = \frac{x_1 - x_0}{t_1 - t_0}$$

and the equation of motion, being linear, is of the form

$$x = a + \left(\frac{x_1 - x_0}{t_1 - t_0}\right)t$$

But when $t=t_0$, $x=x_0$, and hence

$$x_0=a+\left(\frac{x_1-x_0}{t_1-t_0}\right)t_0$$

or

$$a=x_0-\left(\frac{x_1-x_0}{t_1-t_0}\right)t_0$$

and the equation of motion becomes

$$x=x_0+\left(\frac{x_1-x_0}{t_1-t_0}\right)(t-t_0)$$

B. UNIFORMLY ACCELERATED MOTION

36. Properties of uniformly accelerated motion.—From the equation of the motion

$$x=x_0+v_0t+\tfrac{1}{2}\gamma t^2$$

we have

$$v=\frac{dx}{dt}=v_0+\gamma t$$

We observe, then, that γ, being constant, cannot change its sign, while v being linear in t can change its sign *once* only, namely, at the point where

$$v_0+\gamma t=0$$

which corresponds to the instant

$$t_1=-\frac{v_0}{\gamma}$$

and hence to the displacement

$$x_1 = x_0 - \frac{v_0^2}{2\gamma}$$

which gives the unique point in the path beyond which the *mobile* cannot go. The *mobile* can traverse only that portion of its path which lies on one side or the other of this point, the positive side if γ is positive, the negative side if γ is negative, since for $t = \pm\infty$, $x = +\infty$ for $\gamma > 0$ and $x = -\infty$ for $\gamma < 0$.

The general properties of the motion are then contained in the following

THEOREM.—*A uniformly accelerated rectilinear motion has two phases: one previous to the unique instant, where the velocity vanishes and the position M_1 of the mobile is defined by the abscissa* (fig. 16).

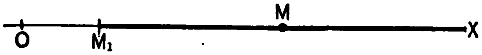


FIG. 16.

$$x_1 = x_0 - \frac{v_0^2}{2\gamma}$$

the other subsequent to this instant.

During the first phase the mobile M travels toward the point M_1 with a uniformly retarded motion and reaches it at the instant

$$t_1 = -\frac{v_0}{\gamma}$$

There it comes to rest and then returns on its path during the second phase, describing the original path in the opposite direction with a uniformly accelerated motion.

It reaches the point M_1 but once, every other point in the path twice. It takes the same time to go from any point M to M_1 as to return and repasses M with the same velocity as at first passage but in opposite direction.

To establish this theorem, we choose the positive direction of the X-axis so as to make γ positive.

Suppose, then, the acceleration positive. Equation [36] may be written

$$x = x_0 - \frac{v_0^2}{2\gamma} + \tfrac{1}{2}\gamma\left(t + \frac{v_0}{\gamma}\right)^2$$

whence
$$\frac{dx}{dt} = \gamma\left(t + \frac{v_0}{\gamma}\right)$$

or, supposing
$$x_1 = x_0 - \frac{v_0^2}{2\gamma}, \qquad t_1 = -\frac{v_0}{\gamma}$$

$$x - x_1 = \tfrac{1}{2}\gamma(t - t_1)^2 = \tfrac{1}{2}\gamma(t_1 - t)^2$$

$$\frac{dx}{dt} = \gamma(t - t_1) = -\gamma(t_1 - t)$$

These equations express the properties indicated. From $t - t_1 = -\infty$ to $t - t_1 = 0$ the *mobile* comes from infinity along the positive axis of X and approaches the point M_1 of abscissa x_1, which it reaches at the instant t_1.

When $t - t_1$ increases from 0 to $+\infty$, $x - x_1$ takes the same values as before, which shows that the *mobile* retraces its path.

For two values of $t - t_1$ equal and of opposite sign, $x - x_1$ takes values equal and of same sign, which shows that the *mobile* takes the same time to go from any point to M_1

as to return; and $\frac{dx}{dx}$ takes values equal and of opposite sign, which shows that the velocities are equal and opposite going and returning.

37. Remark.—We have studied the motion from $t = -\infty$ to $t = +\infty$.

Suppose we wish to know what the motion is during a finite interval, from t_0 to t, say.

If the initial instant t_0 precedes the *critical* instant $t_1 = -\frac{v_0^2}{\gamma}$, that is, if v_0 and γ have opposite signs, then both phases of the motion are comprised in the interval $t - t_0$, and the *mobile* starting from its initial position moves to M_1 and returns to its initial position, which it will pass unless stopped.

38. Equations obtained by taking the space described as the independent variable.—The equation

$$\frac{dx}{dt} = v = v_0 + \gamma t$$

squared gives

$$v^2 = v_0^2 + 2\gamma(v_0 t + \tfrac{1}{2}\gamma t^2)$$

$$= v_0^2 + 2\gamma(x - x_0)$$

The velocity v_0 and the displacement x_0 refer to the epoch $t = 0$. If then h is the space described from the epoch $t = 0$,

$$v^2 = v_0^2 + 2\gamma h \qquad [39]$$

—a fundamental equation.

C. PERIODIC MOTIONS

39. Equation of a periodic motion.—A function $f(t)$ of a variable t is said to be *periodic* when, for a certain positive increment T of the variable t, the function does not change its value, that is, when

$$f(t+T)=f(t) \qquad [40]$$

whatever be t.

If T is the *smallest* increment satisfying this condition, it is called the *period* of the function $f(t)$.

Since [40] is true whatever be t, it follows that

$$f(t\pm nT)=f(t)$$

where n is any positive integer.

Suppose a *mobile* has a motion represented by the equation

$$x=f(t)$$

$f(t)$ being a periodic function of period T. It follows then that:

If the *mobile* at the instant t has the position defined by the abscissa x, it will repass this same position at the end of each successive interval T and have there the same velocity and acceleration.

For $$f(t+T)=f(t)$$

and $$f'(t+T)=f'(t)$$

which shows that the derivative of a periodic function is also periodic. This is true of all the derivatives, and in particular of the two which represent the velocity and acceleration of the *mobile.*

40. Equation of harmonic motion.—A periodic motion which is particularly important in mechanics, and in all branches of physics, is that in which the displacement varies as the sine or cosine of an angle which is itself a linear function of the time.

The equation of such a motion is then, by definition,

$$\left.\begin{aligned} x &= a\cos\theta \\ \text{or}\qquad x &= a\sin\theta \end{aligned}\right\}\qquad \begin{aligned}(1)\\(2)\end{aligned}$$

where θ is a linear function of the time.

Let

$$\theta = nt - \phi$$

where a, n, and ϕ are constants.

By a proper choice of the constants both (1) and (2) define the same motion, hence it is necessary to consider one of them only. In fact if we put

$$\theta_1 = \frac{\pi}{2} - \theta = -nt + \phi + \frac{\pi}{2}$$

(1) takes the form

$$x = a\sin\theta_1$$

where θ_1 is linear with respect to the time.

Again, in (2), which is the one we shall retain, we can always regard n as positive, for if n is negative, then putting

$$\theta_1 = -\theta = -nt + \phi$$

(2) becomes

$$x = a \sin \theta_1$$

in which the coefficient of t is positive. Also, a may be regarded as positive, for if it is negative, then putting

$$a_1 = -a$$

we have

$$x = a_1 \sin \theta_1$$

where a_1 is positive.

Finally ϕ may always be considered to lie between 0 and 2π, since if it is not contained in these limits, it can be brought within them by the addition of a suitable positive or negative integral multiple of 2π, which will not affect the value of x.

We can then write the equation of motion in the form

$$x = a \sin \theta = a \sin(nt - \phi) \qquad [41]$$

where a and u are positive and ϕ lies between 0 and 2π.

41. Amplitude, elongation, period, frequency, argument, phase.—Since $\sin \theta$ varies from -1 to $+1$ for increasing values of t, the displacement x varies between $-a$ and $+a$.

If then on the path $X'OX$ of the *mobile* starting from the origin O we lay off distances

$$OA_0 = OA_1 = a$$

on each side of the origin (fig. 17), the motion consists of an infinite sequence of oscillations of the *mobile* between A_0 and A_1, and is termed a simple vibration or a simple harmonic motion. The length a is called the *amplitude,*

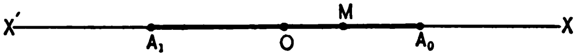


FIG. 17.

and the displacement $OM = x$ at any instant is called the *elongation* of the *mobile* at that instant.

The period T of the motion is the increment which must be given to t to bring x back to its same value in magnitude and sign, and is therefore determined by the relation

$$\sin(nt - \phi) = \sin[n(t+T) - \phi]$$

which gives

$$nT = 2\pi$$

or

$$T = \frac{2\pi}{n}$$

The number of oscillations per unit of time, generally per second, is called the frequency, and the length of one oscillation being T, the frequency is $\frac{1}{T}$, or $\frac{n}{2\pi}$.

The variable angle θ is called the argument, and the constant angle ϕ the phase.

Putting $t = 0$ in [41], we have

$$x_0 = a \sin(-\phi)$$

Whence $-\phi$ is the value of θ at the initial instant.

42. Velocity and acceleration.—The velocity and acceleration are given by

$$v=\frac{dx}{dt}=an\cos\theta=an\cos(nt-\phi) \qquad [42]$$

$$j=\frac{d^2x}{dt^2}=-an^2\sin\theta=-n^2x \qquad [43]$$

HENCE, *the acceleration is proportional to the elongation of the mobile and is constantly directed toward the centre of the path.*

43. Discussion of the motion.—Suppose the phase ϕ to be zero, which is equivalent to supposing that t is taken equal to zero at the instant when the *mobile* passes O. The equation of motion then reduces to

$$x=a\sin nt=a\sin\frac{2\pi t}{T}$$

Whence $$v=\frac{dx}{dt}=an\cos nt=n\sqrt{a^2-x^2}$$

$$j=\frac{d^2x}{dt^2}=-an^2\sin nt=-n^2x$$

It is sufficient to discuss an oscillation, that is, to discuss the motion from $t=-\frac{T}{2}$ to $t=+\frac{T}{2}$, since after each period the motion is reproduced.

From $t=-\frac{T}{4}$ to $t=+\frac{T}{4}$ the elongation varies from $-a$ to $+a$ and the *mobile* describes its complete path in the positive direction from A_1 to A_0. At A_1 when $x=-a$, the velocity is zero, and increases from this point to O, where it has its maximum value na. From A to A_0 it decreases and becomes zero again at A_0. The *mobile* comes instantaneously to rest at A_0 and then describes its path A_0A_1 in the negative direction.

We observe finally that during a complete oscillation the *mobile* passes twice each point of its path, once in the positive direction and once in the negative.

If θ is the argument for which it passes a given point the first time, the argument for which it passes it the second time is

$$\theta_1=\pi-\theta$$

If then it passes the point at the instant t, it again passes it at the instant t_1 determined by

$$\frac{2\pi t_1}{T}=\pi-\frac{2\pi t}{T}$$

or

$$t_1=\frac{T}{2}-t$$

The expression for the velocity

$$\frac{dx}{dt}=an\cos nt=an\cos\frac{2\pi t}{T}$$

shows that at the instant t_1 it has the same velocity as at the instant t, but with the sign changed.

44. Influence of the phase.—We have supposed, in the discussion of the motion, the phase ϕ to be zero. But the motion retains the same properties whatever the phase. Imagine two *mobiles* with the same simple harmonic motion, but different phases, that is, with the same amplitude and period, but one with phase zero and the other with phase ϕ.

Representing by x and x_1 the elongations of the two *mobiles* at the instant t, we have

$$x = a \sin nt$$

$$x_1 = a \sin (nt - \phi)$$

Putting $t_0 = \frac{\phi}{n}$, the last equation may be written

$$x_1 = a \sin n(t - t_0)$$

where ϕ is the phase measured in *angle* and t_0 the phase measured in *time*.

If then for the second *mobile* we measure the time from the instant

$$t_0 = \frac{\phi}{n}$$

its equation of motion is the same as that of the first mobile, which shows that its motion is the same as that of the first, but it passes each point of their common path $t_0 = \frac{\phi}{n}$ units of time after the first.

The two *mobiles* behave as two trains which run between the same two cities, the second t_0 units of time after the first. They have the same motion and stop at the same stations, but one always a fixed interval of time ahead of the other.

CHAPTER IV

RELATIVE MOTION ALONG A LINE

45. Relative motion.—46. Equation of relative motion.—47. Relative velocity and acceleration.—48. Apparent motion.—49. Convective motion.—50. Simultaneous motions.—51. Composition of harmonic motions of same period.—52. Representation of the amplitude and phase by a vector.—53. Equation of the resultant motion.—54. Vibrations of different periods.

45. Relative motion.—Consider the motion of two *mobiles* along the same line $X'OX$ (fig. 18). Let M_c and M_a represent the two *mobiles* and x_c and x_a their abscissas, or displacements, referred to the origin O. Denote the distance M_cM_a by ξ_r, and let the sign of ξ_r be determined by the relation

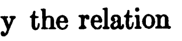


FIG. 18.

$$\xi_r = x_a - x_c \qquad [44]$$

The *algebraic* value of ξ_r then determines the *relative* positions of M_c and M_a, that is, it determines their distance apart and their order on the line $X'OX$, so that *given* the position of one of them the position of the other can be *determined without ambiguity.* ξ_r is termed the *relative displacement of* M_a *with respect to* M_c.

If we are concerned with the *relative* positions only of two *mobiles,* as of two runners or of two race-horses, it is necessary to know ξ_r only. In distinction, x_a and x_c are called the *absolute* displacements of M_a and M_c, and if we require the *absolute* positions of the two runners on the race-track, then x_a and x_c must both be known.

Because of the relation [44] which connects ξ_r, x_a and x_c, it is sufficient to know two of them to determine the third.

The motion of M_a with respect to M_c is termed the *relative* motion of M_a.

46. Equation of relative motion.—If ξ_r be known as a function of the time, then at any instant the relative position of M_a is known. This equation

$$\xi_r = f_r(t)$$

which gives the relative displacement in terms of the time, is known as the *equation of relative motion* of M_a referred to M_c.

If the equations of *absolute* motion of M_a and M_c are known, then the equation of relative motion is at once derivable. For if

$$x_a = f_a(t)$$

and

$$x_c = f_c(t)$$

then

$$\xi_r = f_a(t) - f_c(t) \qquad [45]$$

47. Relative velocity and acceleration.—In exactly the same manner as the notion of absolute velocity was obtained, we can get the notion of relative velocity. The absolute velocity of M_a may be regarded as the rate at which M_a is receding from or approaching O, and the relative velocity of M_a the rate at which it is receding from or approaching M_c. The relative velocity then is the *derivative of* ξ_r *with respect to the time.* Similarly the *relative acceleration* is the derivative of the *relative velocity* with respect to the time.

Differentiating [45], we then have

$$\frac{d\xi_r}{dt}=\frac{dx_a}{dt}-\frac{dx_c}{dt}.$$

or

$$v_r=v_a-v_c \qquad [46]$$

Differentiating a second time,

$$\frac{dv_r}{dt}\equiv\frac{d^2\xi}{dt^2}=\frac{d^2x_a}{dt^2}-\frac{d^2x_c}{dt^2}$$

or

$$j_r=j_a-j_c \qquad [47]$$

HENCE, *the relative displacement, velocity, and acceleration of a mobile* M_a *referred to a second mobile* M_c *is equal to the absolute displacement, velocity, and acceleration of* M_a *minus the absolute displacement, velocity, and acceleration of* M_c.

48. Apparent motion.—Imagine an observer situated on M_c and observing the motion of M_a. The quantities which he would measure would be exactly the relative displacement, velocity, and acceleration of M_a, and he would therefore know only the relative motion of M_a. For this reason relative motion is important and is often termed *apparent motion.*

Suppose, for example, one train to follow another. A passenger on the second train can observe whether he is gaining or losing on the first, and if he has means for measuring the distance between the two trains, can tell how much he is gaining or losing per minute, that is, he can determine the relative or apparent motion of the. first train.

49. Convective motion.—Transposing the terms of [46] and [47], he have

$$v_a = v_c + v_r \qquad [48]$$

$$j_a = j_c + j_r \qquad [49]$$

The absolute velocity and acceleration of M_a are then seen to be made up of two terms, one due to the motion of M_c itself and the other due to the relative motion of M_a referred to M_c. If we imagine M_a to be invariably connected with M_c and carried along with it in its motion, the absolute velocity and acceleration of M_a are the same as those of M_c, but if M_a moves with respect to M_c, an additional term is introduced due to this relative motion. We then divide the absolute motion of M_a into two components, one due to the motion of M_c itself and the second due to the relative motion of M_a referred to M_c. This first component due to the motion of M_c is called the convective motion of M_a.

HENCE, *the absolute* $\left\{\begin{matrix}\textit{velocity}\\ \textit{acceleration}\end{matrix}\right\}$ *of* M_a *equals the sum of the convective* $\left\{\begin{matrix}\textit{velocity}\\ \textit{acceleration}\end{matrix}\right\}$ *and the relative* $\left\{\begin{matrix}\textit{velocity}\\ \textit{acceleration}\end{matrix}\right\}$ of M_a.

Suppose, for example, a man walking down the aisle of a moving car. His absolute velocity—referred to one of the stations as origin, say—is equal to the sum of the absolute velocity of the car and his relative velocity in the car.

A man owns a rifle that gives a velocity of a mile a second to the bullet. He stands on the rear platform of a train moving a mile a minute and fires down the track. What is the motion of the bullet?

Considering the direction in which the car is moving as positive, then the convective velocity of the bullet is

$$v_c = 1 \frac{\text{mile}}{\text{minute}}$$

and its relative velocity is

$$v_r = -60 \frac{\text{miles}}{\text{minutes}}$$

Hence its absolute velocity down the track is

$$v_a = 1 - 60 = -59 \frac{\text{miles}}{\text{minutes}}$$

the minus sign indicating that it is moving in a direction opposite to that of the train.

50. Simultaneous motions.—Due to the method just described, of breaking up the absolute motion into the convective and relative motion, a *mobile* is said to possess both of these motions *simultaneously*, and this way of speaking is justified by what precedes. It is to be understood, however, that it is only an abbreviation of language. The method given for obtaining the absolute motion from these two simultaneous motions is termed the *composition of motions.*

Any number of intermediate *mobiles* $M_{c_1}, M_{c_2} \dots M_{c_\nu}$ may be imagined and the relative motion of each may be given with respect to the one that precedes it. Thus the motion of M_{c_1} may be given with respect to O and the motion of M_{c_2} with respect to $M_{c_1} \dots$ the motion of M_{c_ν} with respect to $M_{c_{\nu-1}}$, and finally the motion of M_a with respect to M_{c_ν}. We then have

$$x_a = x_c + \xi_{12} + \xi_{23} + \dots + \xi_{(\nu-1)\nu} + \xi_r \qquad [50]$$

where $\xi_{(i-1)i}$ denotes the displacement of M_{c_i} relative to $M_{c_{i-1}}$, and ξ_r the displacement of M_a relative to M_{c_ν}.

Differentiating

$$v_a = v_c + v_{r_{12}} + v_{r_{23}} + \dots + v_{r_{(\nu-1)\nu}} + v_r \qquad [51]$$

$$j_a = j_c + j_{r_{12}} + j_{r_{23}} + \dots + j_{r_{(\nu-1)\nu}} + j_r \qquad [52]$$

51. Composition of harmonic motions of same period.—If we have

$$\begin{aligned} x_c &= a_c \sin (nt - \phi_c) \\ \xi_{12} &= a_{12} \sin (nt - \phi_{12}) \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \xi_{(\nu-1)\nu} &= a_{(\nu-1)\nu} \sin (nt - \phi_{(\nu-1)\nu}) \\ \xi_r &= a_r \sin (nt = \phi_r) \end{aligned}$$

then

$$x_a = a_c \sin (nt - \phi_c) + \dots + a_r \sin (nt - \phi_r) \qquad [53]$$

52. Representation of the amplitude and phase by a vector.—In order to study this *resultant* motion of M_a it is convenient to make the following geometric construction. Draw in a plane a line OP (fig. 19) not necessarily in any way connected with the line along which the harmonic motions take place.

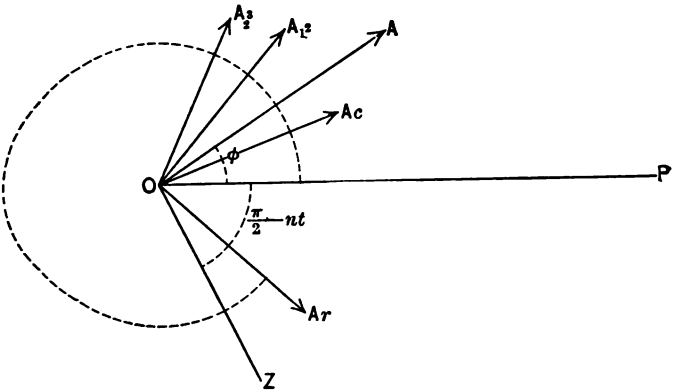


Fig. 19.

From O as origin draw vectors

$$OA_c \ldots\ldots\ldots\ldots OA_{(i-1)i} \ldots\ldots\ldots\ldots OA_r$$

of lengths

$$a_c \ldots\ldots\ldots\ldots a_{(i-1)i} \ldots\ldots\ldots\ldots a_r$$

making angles

$$\phi_c \ldots\ldots\ldots\ldots \phi_{(i-1)i} \ldots\ldots\ldots\ldots \phi_r$$

with the line OP, measured in the positive direction.

53. Equation of the resultant motion.—Construct the geometric sum of these vectors

$$\overline{OA}=\overline{OA}_c+\ \ldots\ +OA_{(i-1)i}+\ \ldots\ +OA_r$$

and let a represent its length and ϕ the angle it makes with OP, measured in the positive direction.

Draw an axis OZ making the angle

$$\frac{\pi}{2}-nt$$

with OP, measured in the *negative* direction.

Then the projection of OA on OZ equals the algebraic sum of the projections of

$$OA_c.\ldots\ldots\ldots\ldots.OA_{(i-1)i}.\ldots\ldots\ldots\ldots.OA_r$$

on OZ for all values of t, and expressing this analytically, we have

$$a\cos\left(\frac{\pi}{2}-nt+\phi\right)=a_c\cos\left(\frac{\pi}{2}-nt+\phi_c\right)+\ \ldots$$
$$+a_{(i-1)i}\cos\left(\frac{\pi}{2}-nt+\phi_{(i-1)i}\right)+\ \ldots\ +a_r\cos\left(\frac{\pi}{2}-nt+\phi_r\right)$$

which may be written

$$a\sin(nt-\phi)=a_c\sin(nt-\phi_c)+\ \ldots\ +a_{(i-1)i}\sin nt-\phi_{(i-1)i})$$
$$+\ldots\ +a_r\sin(nt-\phi_r)$$

Comparing this equation with [53] we see that

$$x_a=a\sin(nt-\phi) \qquad [54]$$

THEREFORE, *the resultant motion of any number of simultaneous harmonic motions of the same period and along the same line is a unique harmonic motion of the same period as the component motions and whose amplitude and phase are given by the geometric sum of the vectors representing the amplitudes and phases of the component motions.*

COROLLARY.—*The relative motion of two mobiles having simple harmonic motions along the same line of same period is a simple harmonic motion with this period, whose amplitude and phase are represented by the geometric difference of the two vectors representing the amplitudes and phases of the component motions.*

54. Vibrations of different periods.—The problem of the composition of two harmonic motions of different periods is much more complex. Suppose, for example, that we wish to find the resultant motion of two harmonic vibrations of periods

$$\frac{2\pi}{n} \quad \text{and} \quad \frac{2\pi}{n_1}$$

its equation is of the form

$$x = a \sin (nt - \phi) + a_1 \sin (n_1 t - \phi_1) \qquad [55]$$

and although each component is harmonic, the resultant motion will be neither harmonic nor, in general, periodic.

It will be periodic but not harmonic when the two periods are commensurable, for if $2\tau\pi$ be the common measure and we put

$$n = k\frac{2\pi}{\tau} \qquad n_1 = k_1\frac{2\pi}{\tau}$$

where k and k_1 are integers, then when t is increased by τ the arguments of the two terms of [55] increase by

$$2k\pi \quad \text{and} \quad 2k_1\pi$$

respectively, so that τ is a period.

If we wish to find the limits of elongation by equating the velocity

$$v = na \cos (nt - \phi) + n_1 a_1 \cos (n_1 t - \phi_1)$$

of the resultant motion to zero, we have the transcendental equation

$$na \cos (nt - \phi) + n_1 a_1 \cos (n_1 t - \phi_1) = 0$$

which can only be solved by approximation or graphically.

CHAPTER V

VELOCITY AND ACCELERATION IN CURVILINEAR MOTION

A. Velocity

55. Displacement.—56. Velocity.—57. Algebraic value of the velocity.—58. Projection of the velocity on the displacement.—59. Geometric representation of the path.—60. Projection of the velocity on any axis.

B. Acceleration

61. Acceleration.—Hodograph.—62. Projection of the acceleration on any axis.—63. Projection of the acceleration on the tangent and principal normal.—64. Projection of the acceleration on the binormal.—65. Condition that the acceleration be constantly tangential or normal to the path. Condition that the acceleration be constantly zero.—66. Composition of motions along the same path.

A. Velocity

55. Displacement.—In rectilinear motion the abscissa of the *mobile* was spoken of as its displacement. This notion extended to motion in any path leads to the definition of the *displacement* of the *mobile* M with respect to the fixed point O (fig. 20) as the vector OM.

Denote this vector by the letter r. The displacement is, then, a variable vector with an arbitrarily chosen origin.

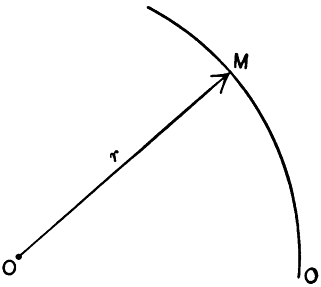


Fig. 20.

56. Velocity.—The velocity in curvilinear motion is defined as the geometric derivative of the displacement, or

$$\overline{V}=\frac{\overline{Dr}}{\overline{Dt}} \qquad [56]$$

The *mobile* being at M at the instant t, and at M_1 at the instant $t+\Delta t$ (fig. 21), then

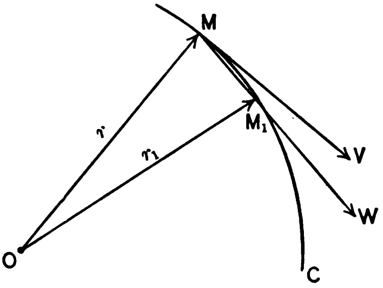


FIG. 21.

$$\overline{V}=\lim \left.\frac{\overline{r}_1-\overline{r}}{\Delta t}\right]_{\Delta t=0}$$

$$=\lim \overline{MW}\Big]_{\Delta t=0}$$

$$=\overline{MV}$$

where MW is a vector in the direction MM_1 of length

$$\frac{MM_1}{\Delta t}$$

and MV is the limiting position of MW as Δt tends to zero, and is therefore tangent to the path C at M.

HENCE, *the direction of the velocity is along the tangent to the path at M, in the direction of motion.*

Now [56] may be written *algebraically*

$$V=\lim \frac{\text{chord } MM_1}{\Delta t}\Big]_{\Delta t=0}$$

But
$$\frac{\text{chord } MM_1}{\Delta t}=\frac{\text{chord } MM_1}{\text{arc } MM_1}\cdot\frac{\text{arc } MM_1}{\Delta t}$$

and therefore

$$V=\lim\left(\frac{\text{chord } MM_1}{\text{arc } MM_1}\cdot\frac{\text{arc } MM_1}{\Delta t}\right)_{\Delta t=0}$$

$$=\lim \frac{\text{chord } MM_1}{\text{arc } MM_1}\cdot\lim \frac{\text{arc } MM_1}{\Delta t}\Big]_{\Delta t=0}$$

$$=\lim \frac{\text{arc } MM_1}{\Delta t}\Big]_{\Delta t=0}$$

since
$$\lim \frac{\text{chord } MM_1}{\text{arc } MM_1}\Big]_{M_1=M}=1.$$

The algebraic value of the velocity may then be regarded as *the limit of the ratio of the distance travelled by the mobile to the time required to travel this distance, as the time tends to zero.*

This is analogous to the velocity in rectilinear motion.

57. Algebraic value of the velocity.—If the equation of motion along the *given* path c be

$$s=\phi(t)$$

where s is the length of the arc M_0M (fig. 22) measured

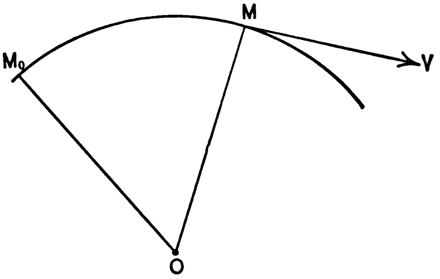


FIG. 22.

from some fixed point M_0 on the path, then the algebraic value of the velocity is

$$V=\frac{ds}{dt}=\phi'(t) \qquad [57]$$

for
$$V=\lim \left.\frac{\text{arc } M_0M}{\Delta t}\right]_{\Delta t=0}=\frac{ds}{dt}$$

58. Projection of the velocity on the displacement.—The projection of V on r is the projection of the geometric derivative of a vector on the vector itself, and hence by [26]

$$V_r=\frac{dr}{dt}$$

If then the equation of motion along the given path be written

$$r=f(t)$$

$$V_r=\frac{dr}{dt}=f'(t) \qquad [58]$$

59. Geometric representation of the path.—The displacement r will describe a cone with vertex at O, and the path c will be a curve lying on this cone.

It may be imagined as traced by a *mobile* which moves along r with the velocity $\frac{dr}{dt}$ while r describes the cone. If $\frac{dr}{dt}$ is zero, the path is the intersection of the cone with a sphere of radius r.

60. Projection of the velocity on any axis.—From the theorem of Art. 12 it follows that

The projection of the velocity on any axis through O, as OX (fig. 23), is the algebraic derivative of the projection of the displacement r on OX.

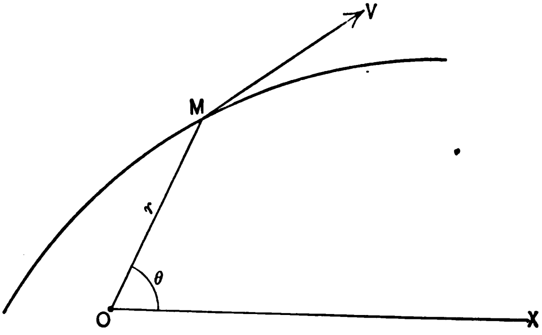


FIG. 23.

Hence

$$V_x = \frac{d}{dt}(r\cos\theta)$$

or

$$V_x = \frac{dr}{dt} - r\sin\theta\frac{d\theta}{dt} \qquad [59]$$

B. ACCELERATION

61. Acceleration.—If the velocity is not constant in magnitude and direction, it is necessary, in order to determine the motion, to know how it varies.

The geometric derivative of the velocity is defined to be the *acceleration*. The acceleration is then a vector MJ (fig. 24), the geometric derivative of the vector MV.

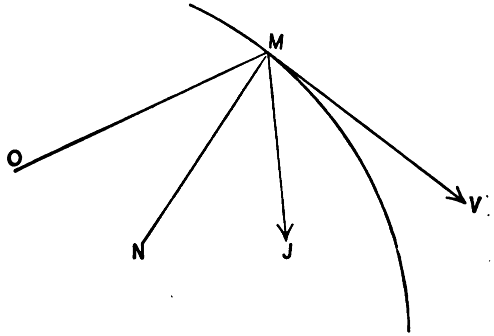


FIG. 24.

We may then write

$$\left.\begin{aligned} \overline{J} &= \frac{\overline{DV}}{Dt} \\ &= \frac{\overline{D^2r}}{Dt^2} \end{aligned}\right\} \qquad [60]$$

Hodograph.—If from a fixed point, O say, a vector be drawn equipollent to MV, the path described by the extremity of this vector is termed the *hodograph* of the motion.

HENCE, *the velocity of the point which describes the hodograph is equipollent to the acceleration of the mobile.*

62. Projection of the acceleration on any axis.—Since (Art. 12) the projection of the geometric derivative of a moving vector on any line is the algebraic derivative of the projection of the vector on that line, it follows that

The projection of the acceleration on any line is the algebraic derivative of the projection of the velocity on that line, and is therefore the second algebraic derivative of the projection of the displacement on that line.

63. Projection of the acceleration on the tangent and principal normal.—Since the projection of the velocity on the tangent at M is

$$\frac{ds}{dt}$$

then the projection of the acceleration on this tangent is (fig. 25)

$$J_t = \frac{d^2s}{dt^2} \qquad [61]$$

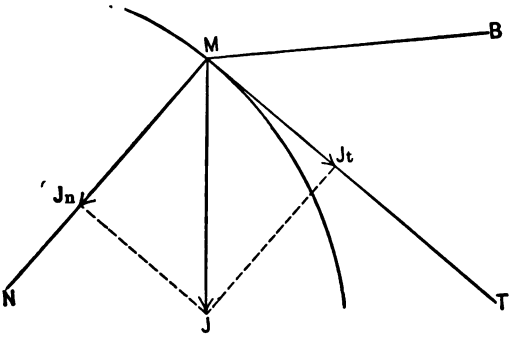


FIG. 25.

Since the principal normal, MN, is perpendicular to the vector OV; then (Art. 12, Corollary) the projection of J on the principal normal, taking the *inward* direction as positive, is

$$J_n = -V\left(\frac{d\phi}{dt}\right)_{\phi=90^\circ}$$

where ϕ is the angle between MN and OV.

This may be written

$$J_n = -V\left(\frac{d\phi}{ds}\cdot\frac{ds}{dt}\right)_{\phi=90^\circ}$$

$$=\frac{V^2}{\rho} \qquad [62]$$

where ρ is the *radius of curvature* at M, since

$$\left(\frac{d\phi}{ds}\right)_{\phi=90^\circ}=\frac{1}{\rho} \text{ and is negative.}$$

64. Projection of the acceleration on the binormal.—Let ψ be the angle between the binormal MB and OV. Then the projection J on MB is

$$J_b = -V\left(\frac{d\psi}{dt}\right)_{\psi=90^\circ}$$

$$= -V\left(\frac{d\psi}{ds}\cdot\frac{ds}{dt}\right)_{\psi=90^\circ}$$

$$=0$$

since $$\left(\frac{d\psi}{ds}\right)_{\psi=90^\circ}=0$$

HENCE, *the acceleration lies in the osculating plane and is directed toward the concave side of the path.*

It is, therefore, as drawn in fig. 25.

65. Condition that the acceleration be constantly tangential or normal to the path.—THEOREM I.—*In order that the acceleration be constantly tangent to the path it is necessary and sufficient that the motion be rectilinear.* For by the first problem of Art. 14 the orientation of the velocity must be constant.

THEOREM II.—*The necessary and sufficient condition that the acceleration be constantly normal to the path is that the motion be uniform.*

For by problem two of Art. 14 the velocity must be constant in magnitude.

Condition that the acceleration be constantly zero.—THEOREM III.—*In order that the acceleration be constantly zero it is necessary and sufficient that the motion be uniform rectilinear.*

For by problem three of Art. 14 the velocity must be constant both in orientation and magnitude.

66. Composition of motions along the same path.—All that has been said in Chapter III on uniform, uniformly accelerated, and periodic motions in a straight line applies here by changing x into s and *acceleration* into *tangential acceleration.*

Similarly, all that has been said concerning relative motion along the same straight line in Chapter IV applies here to relative motion along the same path by making the same changes.

CHAPTER VI

ANGULAR AND AREAL MOTION. EQUATIONS AND GENERAL THEOREMS

67. Angular displacement.—68. Angular velocity.—69. Relation between linear and angular velocities.—70. Angular acceleration.—71. Relation between linear and angular accelerations.—72. Areal velocity.—73. Extension of theorems already found.—74. Equations of motion. Remark. Units of time and displacement. Homogeneity.—75. Equation of uniform motion.—76. Equation of uniformly accelerated angular motion.—77. Equation of motion when angular acceleration is not constant.—78. Properties of uniformly accelerated angular motion.—79. Periodic angular motion. Harmonic motion.—80. Relative angular motion.—81. Composition of harmonic angular motions of same period.

67. Angular displacement.—Consider the case where the path of the *mobile* is a plane curve, and let O (fig. 26) be any point in the plane of the path and OX a fixed axis in that plane.

Let the mobile be at M at the instant t, and denote the angle XOM by θ.

The angular displacement of the *mobile* about O at the instant t is a vector $O\theta$ perpendicular to the plane, of length θ, and so drawn that an observer standing with his feet at O and head at θ sees the mobile moving from *left to right* in its path.

68. Angular velocity.—The *angular velocity* of the *mobile* about O is, by definition, the geometric derivative of the

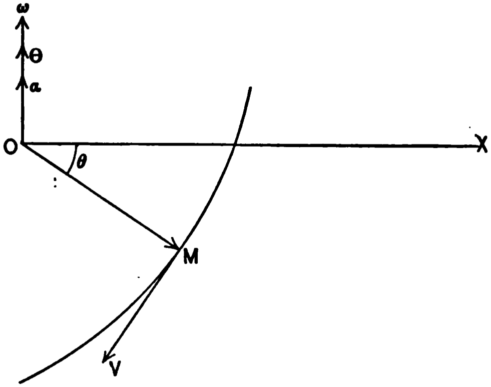


FIG. 26.

angular displacement, and denoting the angular velocity by ω, we have

$$\bar{\omega}=\overline{\frac{D\theta}{Dt}} \qquad [63]$$

Since the angular displacement remains perpendicular to the plane of the path, the angular velocity is also perpendicular to this plane, and is equipollent to the vector $O\omega$ drawn at O normal to the plane and of length equal to the arithmetic value of $\frac{D\theta}{Dt}$.

Now since the geometric derivative of the angular displacement is constantly directed along the displacement, its magnitude is equal to its projection on the displacement, which is $\frac{d\theta}{dt}$.

THEREFORE, *the algebraic value of the angular velocity is*

$$\omega=\frac{d\theta}{dt} \tag{64}$$

69. Relation between linear and angular velocities.—Making use of the theorem of Art. 12, we have at once

$$V\cos\phi=\frac{ds}{dt}\cos\phi=\frac{d}{dt}(r\cos\theta) \tag{65}$$

where ϕ is the angle between the linear velocity and OX.

COROLLARY.—If the path is a circle with centre at O, then $\phi=\theta+\frac{\pi}{2}$, and r is constant, whence

$$-\frac{ds}{dt}\sin\theta=r\sin\theta\frac{d\theta}{dt}$$

or

$$V=\frac{ds}{dt}=r\frac{d\theta}{dt}=r\omega \tag{66}$$

HENCE, *the algebraic value of the linear velocity equals the radius of the circle times the algebraic value of the angular velocity.*

70. Angular accelerations.—The *angular acceleration* is, *by definition,* the geometric derivative of the angular velocity. Denoting it by α, we have

$$\bar{\alpha}=\frac{\overline{D\omega}}{Dt}=\frac{\overline{D^2\theta}}{Dt^2} \tag{67}$$

Since the angular velocity is constantly normal to the plane of the path, the angular acceleration is also, and its algebraic value equals its projection on the angular velocity. Hence

$$\alpha=\frac{d\omega}{dt}=\frac{d^2\theta}{dt^2} \qquad [68]$$

The vector $O\alpha$, of length $\frac{d^2\theta}{dt^2}$, normal to the plane is then equipollent to the angular acceleration.

71. Relation between linear and angular accelerations.—Denoting by ψ the angle between the linear acceleration and velocity, and using the theorem of Art. 12, we have from [65]

$$J\cos\psi=\frac{dV}{dt}=\frac{d}{dt}\left\{\frac{1}{\cos\phi}\frac{d}{dt}(r\cos\theta)\right\} \qquad [69]$$

COROLLARY.—If the path is a circle with centre at O, then

$$V=r\frac{d\theta}{dt}$$

$$\frac{dV}{dt}=\frac{d^2\theta}{dt}$$

and $$J\cos\psi=r\frac{d^2\theta}{dt^2}$$

or $$J_t=r\frac{d^2\theta}{dt^2}=r\alpha \qquad [70]$$

HENCE, *the tangential component of the acceleration of a mobile moving in a circle is equal to the product of the radius and the angular acceleration of the mobile about the centre.*

The normal component is

$$J_n = \frac{V^2}{\rho} = \frac{V^2}{r}$$

and is directed toward the centre.

72. Areal velocity.—Let M_0 be the point where OX intersects the path (fig. 27). The *areal displacement, velocity,* and *acceleration* are defined in exactly the same way as the angular displacement, velocity, and acceleration, except that, instead of the angle θ, the area A of the *curvilinear triangle* M_0OM is used.

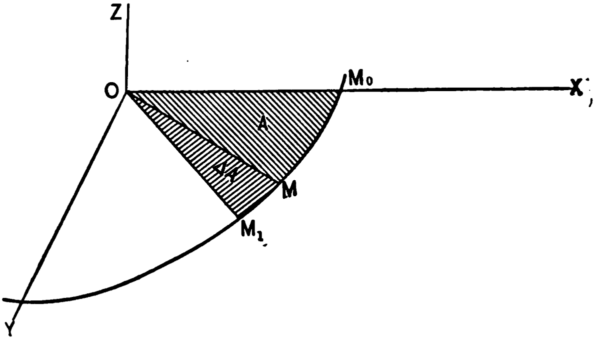


FIG. 27.

Hence $$\text{Areal } V = \frac{dA}{dt}$$

and $$\text{Areal } J = \frac{d \text{ Areal } V}{dt} = \frac{d^2A}{dt^2}$$

Since, when θ and the path are given, A is determined, these can be expressed in terms of the angular velocity and acceleration.

Let ΔA represent the increment of A in time Δt,

then either $$\pi r^2\frac{\Delta\theta}{2\pi} \leqq \Delta A \leqq \pi r_1{}^2\frac{\Delta\theta}{2\pi}$$

or $$\pi r^2\frac{\Delta\theta}{2\pi} \geqq \Delta A \geqq \pi r_1{}^2\frac{\Delta\theta}{2\pi}$$

Whence $$\frac{r^2}{2}\cdot\frac{\Delta\theta}{\Delta t} \leqq \frac{\Delta A}{\Delta t} \leqq \frac{r_1{}^2}{2}\cdot\frac{\Delta\theta}{\Delta t}$$

or $$\frac{r^2}{2}\cdot\frac{\Delta\theta}{\Delta t} \geqq \frac{\Delta A}{\Delta t} \geqq \frac{r_1{}^2}{2}\cdot\frac{\Delta\theta}{\Delta t}$$

Now when Δt tends to zero the two variables between which $\frac{\Delta A}{\Delta t}$ lies tend to the same limit $\frac{r^2}{2}\frac{d\theta}{dt}$

hence $$\frac{dA}{dt} = \frac{r^2}{2}\cdot\frac{d\theta}{dt}$$

or $$\text{Areal } V = \frac{r^2}{2}\cdot\frac{d\theta}{dt} \qquad [71]$$

and $$\text{Areal } J = \frac{d}{dt}\left(\frac{r^2}{2}\cdot\frac{d\theta}{dt}\right) \qquad [72]$$

These formulæ find immediate application in the motion of a planet about the sun, the law of motion being such that the areal acceleration is zero, and hence the areal velocity constant.

To express Areal V and Areal J in terms of the rectangular coordinates of M_1 we have

$$x=r\cos\theta \qquad y=r\sin\theta$$

$$\frac{dx}{dt}=\frac{dr}{dt}\cos\theta-r\sin\theta\frac{d\theta}{dt}$$

$$\frac{dy}{dt}=\frac{dr}{dt}\sin\theta+r\cos\theta\frac{d\theta}{dt}$$

Hence $$x\frac{dy}{dt}-y\frac{dx}{dt}=r^2\frac{d\theta}{dt}$$

Therefore $$\text{Areal } V=\frac{1}{2}\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right) \qquad [73]$$

and $$\text{Areal } J=\frac{1}{2}\left(x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right) \qquad [74]$$

73. Extension of theorems already found. — What has been said in Chapters III and IV about motion along a straight line can for the most part be extended to angular motion about a point by reading " angular displacement," " velocity," and " acceleration " where linear displacement, velocity, and acceleration occur.

74. Equation of motion.—As in the case of rectilinear and curvilinear motion, the equation of the path being supposed *given*, in the form, say,

$$r=F(\theta)$$

the equation of motion may be written

$$\theta=f(t)$$

and if this be known the angular velocity and acceleration are immediately obtained from the relations

$$\omega=\frac{d\theta}{dt}=f'(t)$$

$$\alpha=\frac{d^2\theta}{dt^2}=f''(t)$$

If the angular displacement be taken as the independent variable, the equation of motion may be written in the second form

$$t=\phi(\theta)$$

and the velocity and acceleration are by the same reasoning employed in Art. 33

$$\omega=\frac{1}{\phi'(\theta)}$$

$$\alpha=-\frac{\phi''(\theta)}{\phi'^3(\theta)}$$

Remark: Units of time and displacement. Homogeneity.—The unit of time is, unless otherwise stated, the second, and the unit of angular displacement may be taken as the degree or radian, preferably and usually the latter unless the contrary is stated.

The same remarks as made in Arts. 22 and 23 hold concerning homogeneity and change of units.

75. **Equation of uniform motion.**—As before, the motion is said to be uniform when the angular velocity is constant, and the equation of such motion is

$$\theta=\theta_0+\omega t \qquad [75]$$

obtained by integrating

$$\frac{d\theta}{dt} = \omega, \text{ a constant}$$

The two theorems of Art. 34 now read:

THEOREM I.—*In a uniform angular motion the angular displacement is a linear function of the time.*

THEOREM II.—*Every angular motion whose angular displacement is a linear function of the time is uniform.*

76. Equation of uniformly accelerated angular motion.—By integrating

$$\frac{d^2\theta}{dt^2} = \alpha, \text{ a constant}$$

we arrive at the equation

$$\theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2 \qquad [76]$$

which is the equation of motion when the angular acceleration is constant.

77. Equation of motion when the angular acceleration is not constant.—The discussion of Art. 32 leads at once to the form

$$\theta = f(t, c, c_1)$$

and the same remarks apply.

78. Properties of uniformly accelerated angular motion.—The entire discussion of Arts. 36–38 inclusive will apply with the changes noted above, and give the fundamental equation

$$\omega^2 = \omega_0{}^2 + 2\alpha(\theta - \theta_0) \qquad [77]$$

and the

THEOREM.—*A uniformly accelerated angular motion has two phases: one previous to the unique instant where the velocity vanishes and the position* M_1 *of the mobile is defined by the displacement*

$$\theta_1=\theta_0-\frac{\omega_0^2}{2\alpha}$$

the other subsequent to this instant.

During the first phase the mobile travels toward the point M_1 *with a uniformly retarded angular motion, and reaches it at the instant*

$$t_1=-\frac{\omega_0}{\alpha}$$

There it comes to rest and then returns on its path during the second phase, describing the original path in the opposite direction with uniformly accelerated angular motion.

It reaches the point M_1 *but once, every other point in the path twice. It takes the same time to go from any point* M *to* M_1 *as to return, and repasses* M *with the same angular velocity as at first passage but in the opposite direction.*

79. Periodic angular motion. Harmonic motion.—Harmonic angular motion being defined in the same way as harmonic linear motion, its equation is

$$\theta=\beta\sin(nt-\phi) \qquad [78]$$

and amplitude, elongation, period, frequency, argument, and phase have the same meaning as in Art. 41, where angle is read for distance.

The entire argument of Arts. 42–44 inclusive applies, giving the period

$$T=\frac{2\pi}{n}$$

and frequency

$$\frac{1}{T}=\frac{n}{2\pi}$$

angular velocity and acceleration

$$\omega=\frac{d\theta}{dt}=\beta n\cos(nt-\phi) \qquad [79]$$

$$\alpha=\frac{d^2\theta}{dt^2}=-n^2\theta \qquad [80]$$

The *mobile* vibrates about the centre of rotation O (fig. 28), coming to rest at the points A_0 and A_1 and returning on the path, repassing any point with the same angular velocity but in opposite direction. The phase has the same influence as in rectilinear harmonic motion.

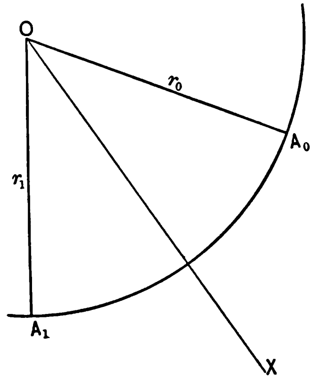


FIG. 28.

80. Relative angular motion.—All the statements and theorems of Chapter IV on relative motion along a line apply to angular motion if the terms be properly interpreted. Suppose two *mobiles* M_c and M_a to be moving about the same centre O (fig. 29), the angle θ_c, which is the absolute angular displacement of M_c, is termed the *convective angular displacement of* M_a, and ϑ_r is termed the *relative angular displacement of* M_a.

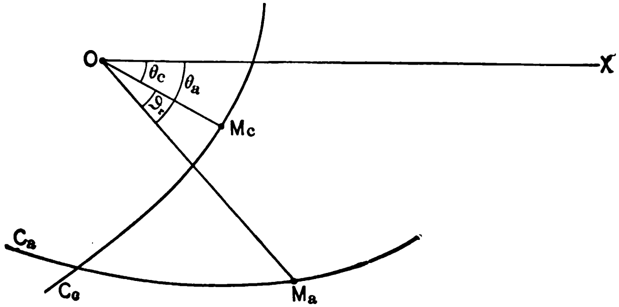


FIG. 29.

We have then

$$\vartheta_r=\theta_a-\theta_c \qquad [81]$$

$$\omega_r=\frac{d\vartheta_r}{dt}=\frac{d\theta_a}{dt}-\frac{d\theta_c}{dt}=\omega_a-\omega_c \qquad [82]$$

$$\alpha_r=\frac{d^2\vartheta_r}{dt^2}=\frac{d^2\theta_a}{dt^2}-\frac{d^2\theta_c}{dt^2}=\alpha_a-\alpha_c \qquad [83]$$

The equation of relative angular motion is

$$\vartheta_r=f_a(t)-f_c(t) \qquad [84]$$

The equations of Art. 50 for the composition of motions become

$$\theta_a=\theta_c+\vartheta_{12}+\ldots+\vartheta_{(\nu-1)\nu}+\vartheta_r \qquad [85]$$

$$\omega_a=\omega_c+\omega_{r_{12}}+\ldots+\omega_{r_{(\nu-1)\nu}}+\omega_r \qquad [86]$$

$$\alpha_a=\alpha_c+\alpha_{r_{12}}+\alpha_{r_{23}}+\ldots+\alpha_{r_{(\nu-1)\nu}}+\alpha_r \qquad [87]$$

81. Composition of harmonic angular motions of same period.—The equations of Arts. 51, 52, and 53 become

$$\begin{aligned}
\theta_c &= \beta_c \sin (nt - \phi_c) \\
\vartheta_{r_{12}} &= \beta_{12} \sin (nt - \phi_{12}) \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\vartheta_{(\nu-1)\nu} &= \beta_{(\nu-1)\nu} \sin (nt - \phi_{(\nu-1)\nu}) \\
\vartheta_r &= \beta_r \sin (nt - \phi_r) \\
\theta_a &= \beta_c \sin (nt - \phi_c) + \ldots + \beta_r \sin (nt - \phi_r) \\
&= \beta \sin (nt - \phi)
\end{aligned}$$

and the same discussion as in Art. 54 applies to composition of angular harmonic motions of different periods.

CHAPTER VII

MOTION REFERRED TO COORDINATE AXES

82. Important remark.—83. Projection of the motion on a plane and on an axis.—84. Projection of the velocity and acceleration.—85. Equations of motion referred to the coordinate axes.—86. Equations of the projected motion on the coordinate planes.—87. Projection of the path.—88. Equations of the projected motion on the coordinate axes.—89. Projections of the velocity.—90. Equations of motion of the point which describes the hodograph.—91. Projections of the acceleration on the coordinate axes.—92. Résumé.—93. Most general motion in which the projected motions are uniformly accelerated.—94. Motion in which the projected motions are harmonic.—95. Rectilinear harmonic motion considered as the projection of uniform circular motion.

82. Important remark.—Throughout this chapter the axes will be assumed rectangular and the projections orthogonal, however, most of the statements and theorems will be true of oblique axes with the necessary modifications.

83. Projection of the motion on a plane and on an axis.—Consider (fig. 30) a *mobile* describing in space any path whatever and a fixed plane Π. Let M be the position of the mobile at the instant t. Project M on Π into m. To each position of M in its path will correspond a determinate point m. The point m may then be regarded as a second *mobile*, whose motion is completely defined by that of M.

The motion of m is termed *the projection of the motion of M on the plane Π.*

Let $X'X$ be a fixed axis—not necessarily in any way connected with the plane Π—and project M on this axis into m'. Regarding this point m' as a *mobile,* its *rectilinear* motion is termed the *projection of the motion of M on the axis X'X.*

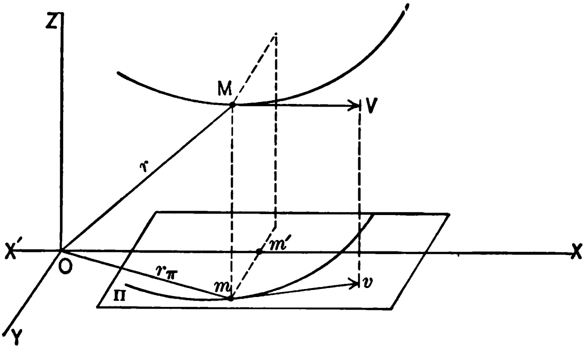


FIG. 30.

84. Projection of the velocity and acceleration.—The velocity of M is the geometric derivative of the displacement r,

or
$$\overline{V}=\frac{\overline{Dr}}{Dt}$$

Hence by the theorem of Art. 12, that the projection of the geometric derivative of a vector on a plane is equipollent to the geometric derivative of the projected vector, we have

$$\overline{V_\pi}=\left(\frac{\overline{Dr}}{Dt}\right)_\pi=\frac{\overline{Dr_\pi}}{Dt}=v \qquad [88]$$

from which follows the

THEOREM I.—*The projection of the velocity on any plane is equipollent to the velocity of the projected motion in that plane.*

Again, from

$$\overline{J}=\frac{\overline{DV}}{Dt}=\frac{\overline{D^2r}}{Dt^2}$$

we have, by applying the theorem of Art. 12,

$$\overline{J_\pi}=\left(\frac{\overline{DV}}{Dt}\right)_\pi=\frac{\overline{Dv}}{Dt}=j \tag{89}$$

whence the

THEOREM II.—*The projection of the acceleration on any plane is equipollent to the acceleration of the projected motion in that plane.*

If instead of projections on the plane Π we consider projections on the axis OX the same reasoning gives.

THEOREM III.—*The projection of the velocity on any axis is equal to the velocity of the projected motion on that axis.*

THEOREM IV.—*The projection of the acceleration on any axis is equal to the acceleration of the projected motion on that axis.*

85. Equations of motion referred to the coordinate axes.—Let x, y, z be the coordinates of the *mobile* at any instant t. Then x, y, and z are uniform, and continuous functions of the time.

Let

$$\left.\begin{aligned} x&=f\ (t)\\ y&=f_1(t)\\ z&=f_2(t)\end{aligned}\right\} \tag{90}$$

These three equations taken together are called the *equations of motion,* since at each instant they define the position of the *mobile* uniquely.

But any two of them together and any one of them have interpretations which it is useful to know.

86. Equations of the projected motion on the coordinate planes.—Suppose the motion projected on the XY plane. The projected *mobile* has in this plane at each instant the same coordinates x and y as the *mobile* in space. Its equations of motion are then

$$x=f(t) \qquad y=f_1(t) \qquad z=0$$

that is, the two equations

$$x=f(t) \qquad y=f_1(t)$$

are the equations of the projected motion on the XY-plane.

Similarly the pairs of equations

$$y=f_1(t) \qquad z=f_2(t)$$

and

$$x=f(t) \qquad z=f_2(t)$$

are the equations of the projected motion in the other two coordinate planes.

87. Projection of the path.—Since the two equations

$$x=f(t) \qquad y=f_1(t)$$

considered in the XY-plane are the equations of the projected motion in that plane the elimination of t gives a relation between x and y satisfied at each instant by the coordinates of the projected *mobile.* *They are, therefore,* *the parametric equations of the path of the projected mobile, and hence of the projected path.*

Similarly the remaining two pairs give the projected path in the other two coordinate planes. Two of these projected paths determine the path in space.

The three equations of motion

$$x=f(t) \qquad y=f_1(t) \qquad z=f_2(t)$$

may also be regarded as the parametric equations of the path in space.

88. Equations of the projected motion on the coordinate axes.—Projecting the *mobile* on the X-axis, the projected *mobile* will have the same abscissa as the *mobile* in space, and its equation of motion is therefore

$$x=f(t)$$

Similarly the remaining two equations

$$y=f_1(t) \qquad z=f_2(t)$$

are the equations of the projected motion on the other two axes.

Hence each equation considered alone is the equation of the projected motion on the corresponding axis.

89. Projections of the velocity.—The velocities in the projected motions on the coordinate axes are

$$\frac{dx}{dt}=f'(t) \qquad \frac{dy}{dt}=f_1'(t) \qquad \frac{dz}{dt}=f_2'(t)$$

But by Theorem III of Art. 84 these are equal to the projections of the velocity of the *mobile* in space. Hence

$$V_x=\frac{dx}{dt}=f'(t) \qquad V_y=\frac{dy}{dt}=f_1'(t) \qquad V_z=\frac{dz}{dt}=f_2'(t) \qquad [91]$$

which give the

THEOREM.—*The projections of the velocity of a mobile on the three coordinate axes equal the first derivatives of the coordinates of the mobile with respect to the time.*

90. Equations of motion of the point which describes the hodograph.—The hodograph is the path described by the extremity of a vector equipollent to the velocity, and drawn from a fixed point, which we may take as O. The coordinates of this extremity are at any instant

$$x_1=V_x \qquad y_1=V_y \qquad z_1=V_z$$

HENCE, *the three equations*

$$x_1=f'(t) \qquad y_1=f_1'(t) \qquad z_1=f_2'(t)$$

may be regarded either as the equations of motion of the point which describes the hodograph, or as the parametric equations of the hodograph itself.

91. Projections of the acceleration on the coordinate axes.—By Theorem IV of Art. 84 we have at once

$$J_x=\frac{d^2x}{dt^2}=f''(t) \quad J_y=\frac{d^2y}{dt^2}=f_1''(t) \quad J_z=\frac{d^2z}{dt^2}=f_2''(t) \qquad [92]$$

These equations give the

THEOREM.—*The projections of the acceleration of a mobile* *on the coordinate axes equal the second derivatives of the coordinates of the mobile with respect to the time.*

92. **Résumé.**—Denoting the first derivatives of the coordinates of a mobile with respect to the time by

$$x' \qquad y' \qquad z'$$

and the second derivatives by

$$x'' \qquad y'' \qquad z''$$

we have the following propositions:

1° The velocity V of a *mobile* M is the geometric sum of the three velocities

$$x' \qquad y' \qquad z'$$

drawn from M parallel to the three axes of coordinates, whence

$$\overline{V} = \overline{x'} + \overline{y'} + \overline{z'} \qquad [93]$$

and

$$V^2 = x'^2 + y'^2 + z'^2 \qquad [94]$$

2° The acceleration J of a *mobile* M is the geometric sum of the three accelerations

$$x'' \qquad y'' \qquad z''$$

drawn from M parallel to the three coordinate axes, whence

$$\overline{J} = \overline{x''} + \overline{y''} + \overline{z''} \qquad [95]$$

and

$$J^2 = x''^2 + y''^2 + z''^2 \qquad [96]$$

3° In projecting these equipollences on a line L whose direction cosines are α, β, γ, we have

$$V_l=\alpha x'+\beta y'+\gamma z' \qquad [97]$$

$$J_l=\alpha x''+\beta y''+\gamma z'' \qquad [98]$$

93. Most general motion in which the projected motions are uniformly accelerated.—Here

$$J_x \qquad J_y \qquad J_z$$

are constants, and integrating

$$x''=J_x \qquad y''=J_y \qquad z''=J_z$$

we have

$$\left.\begin{aligned} x&=a+a't+\tfrac{1}{2}J_xt^2\\ y&=b+b't+\tfrac{1}{2}J_yt^2\\ z&=c+c't+\tfrac{1}{2}J_zt^2 \end{aligned}\right\} \qquad [99]$$

Multiplying these three equations by the arbitrary constants λ, λ_1 and λ_2 adding,

$$\begin{aligned} \lambda x+\lambda_1 y+\lambda_2 z=\lambda a+\lambda_1 b+\lambda_2 c&\\ +(\lambda a'+\lambda_1 b'+\lambda_2 c')t&\\ +\tfrac{1}{2}(\lambda J_x+\lambda_1 J_y+\lambda_2 J_z)t^2& \end{aligned}$$

Choosing λ, λ_1, and λ_2 so that

$$\lambda a'+\lambda_1 b'+\lambda_2 c'=0$$

$$\lambda J_x+\lambda_1 J_y+\lambda_2 J_z=0$$

which can always be done, we have

$$\lambda x+\lambda_1 y+\lambda_2 z=\lambda a+\lambda_1 b+\lambda_2 c \qquad [100]$$

At every instant, then, the coordinates of the *mobile* satisfy an equation of the first degree, and hence *the path lies in the plane defined by equation [100]*.

Again, since J_x, J_y, and J_z are constants, J is constant both in magnitude and direction, and choosing this direction as the Z-axis and the plane of the path as the XZ-plane, the equations of motion are of the *form*

$$\left.\begin{aligned} x&=a+a't\\ y&=0\\ z&=c+c't+\tfrac{1}{2}Jt^2 \end{aligned}\right\} \qquad [101]$$

Eliminating t between the first and third of these,

$$z=c+c'\left(\frac{x-a}{a'}\right)+\tfrac{1}{2}J\left(\frac{x-a}{a'}\right)^2 \qquad [102]$$

which is the equation of the projected path in the XZ-plane, and hence of the path of the *mobile*.

94. Motion in which the projected motions are harmonic.— The equations of motion are

$$\begin{aligned} x&=a\sin(nt-\phi_1)\\ y&=b\sin(nt-\phi_2)\\ z&=c\sin(nt-\phi) \end{aligned}$$

Expanding $\sin(nt-\phi_i)$, we see that each of these equations is linear in sin nt and cos nt. Eliminating these two variables, we have an equation

$$Ax+By+Cz+D=0 \qquad [103]$$

linear in x, y, and z, which shows that *the path is a plane curve.*

Taking this plane as the plane of XY, the equations of motion take the form

$$\left.\begin{aligned} x&=a\sin(nt-\phi_1)\\ y&=b\sin(nt-\phi_2)\\ z&=0 \end{aligned}\right\} \qquad [104]$$

Solving the first two of these equations for sin ut and cos ut and substituting in

$$\cos^2 nt+\sin^2 nt=1,$$

we obtain the equation of an ellipse, which is the path of the *mobile.*

In particular, if the difference of phase is $\frac{\pi}{2}$, equations [104] take the form

$$x=a\sin(nt-\phi)$$

$$y=b\cos(nt-\phi)$$

Whence

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

The projections of the acceleration are

$$J_x=x''=-n^2x$$

$$J_y=y''=-n^2y$$

Whence $$J^2 = -n^4(x^2+y^2) = -n^4r^2$$

and $$J_x = -n^2r\cdot\frac{x}{r} = -J\cdot\frac{x}{r}$$

$$J_y = -n^2r\cdot\frac{y}{r} = -J\cdot\frac{y}{r}$$

HENCE, *the acceleration of the mobile is constantly directed toward the centre of the elliptic path and its algebraic value is*

$$J = -n^2r$$

from which it follows that it is proportional to the distance of the mobile from the centre.

If $b=a$, the path is circular and the acceleration is toward the centre and constant. The tangential acceleration is consequently zero, and the circular motion is uniform.

95. **Rectilinear harmonic motion considered as the projection of uniform circular motion.**—Every harmonic vibration in a straight line may be looked upon as the projection of a certain uniform circular motion on one of its diameters. Suppose the harmonic motion defined by

$$x = a \sin (nt - \phi)$$

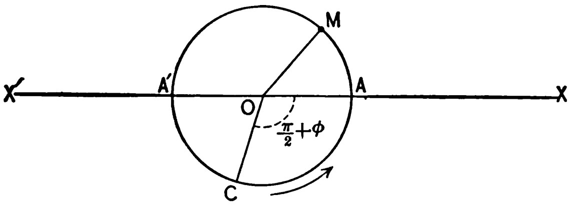


FIG. 31.

along the line $X'OX$ (fig. 31). From O as centre with radius a construct a circle, and imagine this circle traversed by a fictitious *mobile* M with constant angular velocity n and in a direction chosen at will—that of the arrow, say.

A the instant $t=0$ let the fictitious *mobile* M be at C, so that the angle $\widehat{AOC}$ measured in a direction opposite to that of the motion, that is, in the negative direction, equals $\frac{\pi}{2}+\phi$. At any other instant t let the fictitious *mobile* be at M. Then the angle

$$\theta=\widehat{AOM}$$

measured in the positive direction is

$$\theta=nt-\frac{\pi}{2}-\phi$$

and the projection m of the fictitious *mobile* M has the equation of motion

$$\begin{aligned}x&=a\cos\theta=a\cos\left(nt-\frac{\pi}{2}-\phi\right)\\&=a\sin(nt-\phi)\end{aligned}$$

Hence m coincides with the *mobile* making the harmonic vibrations along $X'OX$, which vibrations may therefore be considered as the projection of the motion of M.

CHAPTER VIII

RELATIVE MOTION. MOVING AXES

96. Fixed and moving axes.—97. Equations of absolute and relative motion.—98. Statement of the problem.—99. Motion of the relative axes.—100. Solution of the problem.

96. Fixed and moving axes.—Consider a mobile M describing a path in space, and refer the motion to a set of axes O–XYZ (fig. 32). Imagine a second set of rectangular axes A–$\Xi H Z$ in motion with respect to O–XYZ.

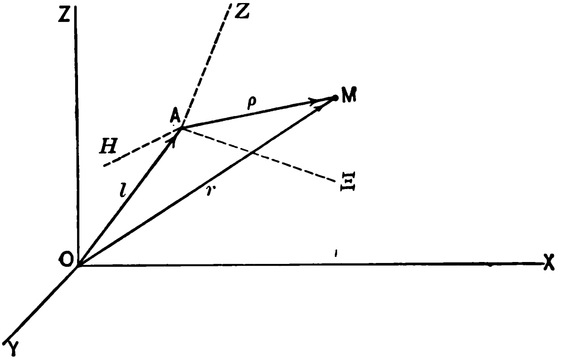


FIG. 32.

The axes O–XYZ are called the *fixed or absolute axes*, and A–$\Xi H Z$ the *moving or relative axes*. The displacement $\mathbf{r}$ and coordinates x, y, z of M referred to the fixed axes are termed the *absolute displacement and coordinates of M*, while the displacement ρ and coordinates ξ, η, ζ of M referred to the moving axes are termed the *relative displacement and coordinates of M*. Finally, let l denote the absolute displacement of A, the origin of the moving axes. The motion and path of M referred to the fixed axes are termed the *absolute motion and path of M*, while the motion and path referred to the moving axes are termed the *relative motion and path.*

97. **Equations of absolute and relative motion.**—The equations of absolute motion are

$$x = f(t) \qquad y = f_1(t) \qquad z = f_2(t) \qquad [105]$$

expressing the absolute coordinates as functions of the time, while the equations of relative motion are

$$\xi = \phi(t) \qquad \eta = \phi_1(t) \qquad \zeta = \phi_2(t) \qquad [106]$$

expressing the relative coordinates as functions of the time.

98. **Statement of the problem.**—The problem to be considered is the following:

1° *Given the motion of the relative axes and the relative motion of the mobile, determine its absolute motion.*

2° *Given the motion of the relative axes and the absolute motion of the mobile, determine its relative motion.*

99. **Motion of the relative axes.**—The motion of the relative axes is known if at every instant the position of the relative *trihedron* A-$\Xi H Z$ is known, and its position is known if the absolute coordinates of A and the absolute direction cosines of the three lines $A\Xi$, AH, and AZ are given as functions of the time.

Denote by o, o_1, o_2 the absolute coordinates of A and let the absolute direction cosines of the relative axes be given by the following *schema:*

	Ξ	H	Z
X	a	b	c
Y	a_1	b_1	c_1
Z	a_2	b_2	c_2

where o, o_1, o_2 and the nine divection cosines $a \ldots c_2$ are supposed *known* functions of the time.

100. Solution of the problem.—*First case.*—The solution consists in this case in finding the *initial* absolute velocity and the absolute acceleration of M in terms of the time.

From the equipollence

$$\bar{r} = \bar{l} + \bar{\rho} = \bar{l} + \bar{\xi} + \bar{\eta} + \bar{\zeta}$$

projected on the three axes

$$\left.\begin{aligned} x &= o + a\xi + b\eta + c\zeta \\ y &= o_1 + a_1\xi + b_1\eta + c_1\zeta \\ z &= o_2 + a_2\xi + b_2\eta + c_2\zeta \end{aligned}\right\} \qquad [107]$$

Differentiating these equations we obtain the projections of the absolute velocity

$$\left.\begin{aligned} x' &= (o' + a'\xi + b'\eta + c'\zeta) + (a\xi' + b\eta' + c\zeta') \\ y' &= (o_1' + a_1'\xi + b_1'\eta + c_1'\zeta) + (a_1\xi' + b_1\eta' + c_1\zeta') \\ z' &= (o_2' + a_2'\xi + b_2'\eta + c_2'\zeta) + (a_2\xi' + b_2\eta' + c_2\zeta') \end{aligned}\right\} \qquad [108]$$

New York Public Library
Webster Branch
538 East 76th Street,
CENTRAL COLLECTION
Circulating Department

Denoting by V_c the vector whose projections on the absolute axes are

$$\left.\begin{aligned} V_{c_x} &= o' + a'\xi + b'\eta + c'\zeta \\ V_{c_y} &= o_1' + a_1'\xi + b_1'\eta + c_1'\zeta \\ V_{c_z} &= o_2' + a_2'\xi + b_2'\eta + c_2'\zeta \end{aligned}\right\} \qquad [109]$$

and by V_r the vector whose projections on the absolute axes are

$$\left.\begin{aligned} V_{r_x} &= a\xi' + b\eta' + c\zeta' \\ V_{r_y} &= a_1\xi' + b_1\eta' + c_1\zeta' \\ V_{r_z} &= a_2\xi' + b_2\eta' + c_2\zeta' \end{aligned}\right\} \qquad [110]$$

we have for the absolute velocity of M the vector V_a given by the equipollence

$$\overline{V}_a = \overline{V}_c + \overline{V}_r \qquad [111]$$

Now V_c is that component of V_a which is obtained by imagining M invariable with respect to the moving axes, while V_r is the component obtained by imagining the relative axes at rest, but the *mobile* to be in motion with respect to them. V_r, then, is the velocity which M would appear to possess to an observer situated on the relative trihedron and carried along with it, while V_c is the velocity which a *mobile* fixed at M and carried alone with the relative trihedron would have due to the motion of this trihedron. V_c is termed the *convective* velocity of M and V_r the *relative* velocity of M.

The equipollence above then gives the

THEOREM.—*The absolute velocity of a mobile is equipollent to the geometric sum of the convective and relative velocity of the mobile.*

Differentiating equations [108]

$$\left.\begin{aligned}
x'' &= (o''+a''\xi+b''\eta+c''\zeta)+(a\xi''+b\eta''+c\zeta'')+2(a'\xi'\\
&\qquad +b'\eta'+c'\zeta')\\
y'' &= (o_1''+a_1''\xi+b_1''\eta+c_1''\zeta)+(a_1\xi''+b_1\eta''+c_1\zeta'')\\
&\qquad +2(a_1'\xi'+b_1'\eta'+c_1'\zeta')\\
z'' &= (o_2''+a_2''\xi+b_2''\eta+c_2''\zeta)+(a_2\xi''+b_2\eta''+c_2\zeta'')\\
&\qquad +2(a_2'\xi'+b_2'\eta'+c_2'\zeta')
\end{aligned}\right\}\qquad [112]$$

Denoting by J_c the vector whose projections on the absolute axes are

$$\left.\begin{aligned}
J_{c_x} &= o''+a''\xi+b''\eta+c''\zeta\\
J_{c_y} &= o_1''+a_1''\xi+b_1''\eta+c_1''\zeta\\
J_{c_z} &= o_2''+a_2''\xi+b_2''\eta+c_2''\zeta
\end{aligned}\right\}\qquad [113]$$

by J_r the vector whose projections are

$$\left.\begin{aligned}
J_{r_x} &= a\xi''+b\eta''+c\zeta''\\
J_{r_y} &= a_1\xi''+b_1\eta''+c_1\zeta''\\
J_{r_z} &= a_2\xi''+b_2\eta''+c_2\zeta''
\end{aligned}\right\}\qquad [114]$$

and by J_t the vector whose projections are

$$\left.\begin{aligned}
J_{t_x} &= 2(a'\xi'+b'\eta'+c'\zeta')\\
J_{t_y} &= 2(a_1'\xi'+b_1'\eta'+c_1'\zeta')\\
J_{t_z} &= 2(a_2'\xi'+b_2'\eta'+c_2'\zeta')
\end{aligned}\right\}\qquad [115]$$

we have for the absolute acceleration of M the vector J_a given by the equipollence

$$\overline{J_a}=\overline{J_c}+\overline{J_r}+\overline{J_t}\qquad [116]$$

Now J_c is the component of the absolute acceleration due to the motion of the axes, that is, it is what the absolute acceleration would become if the *mobile* were invariable with respect to the relative axes, and J_r is the component due to the relative motion, that is, it is what the absolute acceleration would become if the relative axes were at rest, and therefore J_c is the *convective* and J_r the *relative* acceleration of the *mobile.* The third component is termed the *turning* acceleration. The equipollence above, then, gives the

THEOREM.—*The absolute acceleration of a mobile is equipollent to the geometric sum of the convective acceleration, the relative acceleration, and the turning acceleration.*

REMARK.—The turning acceleration depends only on the velocities of the relative axes and the *mobile,* and hence vanishes when the relative axes are at rest, and when the *mobile* is relatively at rest, but both the convective and the relative acceleration may vanish and still leave the turning acceleration not zero. It will, however, then be constant.

Second case.—In this case the initial relative velocity and the relative acceleration as a function of the time is required. These are at once obtained by solving equations [111] and [116] for V_r and J_r, giving

$$V_r = V_a - V_c \qquad [117]$$

$$J_r = J_a - J_c - J_t \qquad [118]$$

the components of the relative velocity required in calculating J_t being obtained by projecting [117] on the relative axes.

CHAPTER IX

MECHANICS OF A FREE PARTICLE

101. Material point or particle.—102. Purpose of this book.—103. The problem of mechanics.—104. The rôle of observation and experiment.—105. Principles of mechanics.—106. The absolute axes.—Isolated particle.—107. First principle of mechanics.—108. Meaning of the first principle.—109. Field of force.—110. Uniform field of force.—111. Constant field of force.—112. Superposition of fields of force.—113. Second principle of mechanics.—114. Meaning of the second principle.—115. Reaction of the particle on the field.—116. Third principle of mechanics.—117. Meaning of the third principle.—118. Properties of the coefficients ρ_{ij}.—119. Definition of mass.—120. Force.—121. Observations on the notion of force.—122. Composition of forces. Resultant.—123. Decomposition of forces. Components.—124. Equilibrium of a free particle.—125. Force acting on a particle at rest.—126. New definition of direction of force.—127. Tangential and normal force.—128. Resistance.

101. **Material point or particle.**—In the preceding chapters we have studied the motion of a *geometric point*, and have seen that for the complete determination of this motion only the notions of space and time are necessary. Knowing at any instant the displacement and velocity of the point and knowing the acceleration as a function of the time we can assign the motion completely. To do this it is only necessary to integrate the differential equations of motion of the point

$$\left.\begin{aligned} x'' &= J_x \\ y'' &= J_y \\ z'' &= J_z \end{aligned}\right\} \qquad [119]$$

under the restrictions that the displacement and velocity shall have the assigned values at the given instant, which restrictions serve to determine the constants of integration. To effect the integration and obtain the equations of motion in their finite form is a problem of pure mathematics, which, while theoretically possible, may be of great practical difficulty. However, with it as such we have not to do, the problem, considered as a problem of kinematics, being solved when we have obtained the differential equations of the motion.

As soon, however, as we begin to apply the theorems of kinematics to the material universe we meet the difficulty that the geometric point has no existence in the physical world, and in order to discuss the motion of bodies as they exist we must make certain assumptions concerning them.

We therefore place ourselves under so-called ideal conditions and assume a portion of matter which we conceive as concentrated at a geometric point. It is with this *material point* or *particle* that we deal in theory, and in applications we neglect the dimensions of the actual portions of matter under consideration, thereby obtaining a more or less close approximation to the true motion. Certain extended bodies, spheres for example, behave under *certain* conditions *exactly* as a *material point* situated at their centre, so that we obtain their true motions.

102. Purpose of this book.—It is my purpose here to develop in a rigorous manner the elementary theory of the motion of a *material point* or *particle*, and to thereby furnish a point of departure for the consideration of the motion of bodies as they actually occur in the material universe. This elementary exposition will then serve as an introduction to that portion of the subject which is known as *rational mechanics,* and in which the mathematical theory of the motion of portions of matter of *ideal* forms is investigated under *ideal* conditions, leaving the special applications to the particular sciences.

103. The problem of mechanics.—Since, as we have seen, the motion of a geometric point is known as soon as we know its *initial displacement* and *velocity* and its *acceleration as a function of the time,* so the motion of a particle is known as soon as we can assign the displacement and velocity at a *particular instant* and the acceleration at *every instant.*

In kinematics the acceleration was given as one of the known quantities of the problem, but in mechanics, on the other hand, the particle and the *physical conditions* under which the motion takes place are given, and the problem is:

To determine the law of the acceleration, and hence the motion.

The converse problem may also be proposed:

Determine what physical conditions will produce a required motion.

We have here an essentially different and very much larger problem than was discussied in kinematics. There we could assign to the *mobile* any displacement, velocity, and acceleration we pleased and determine what motion would take place, or we could imagine the mobile to have any motion we pleased and investigate the manner in which the displacement, velocity, and acceleration varied along the path. In mechanics, on the contrary, we *find the particle under certain physical conditions,* and its subsequent motion is entirely independent of us; we can only investigate the manner in which it actually takes place.

104. The rôle of observation and experiment.—For this we must have recourse to observation and experiment. We can to some extent control the conditions under which a particle shall move, but for the most part we are reduced to observation both on the conditions and the resulting motions. The motion of bodies on the Earth furnishes some data, but by far the most important knowledge is obtained by observing the motions of the bodies of the solar system about each other.

105. Principles of mechanics.—The results of these observations have been reduced to three independent statements, known as the *principles of mechanics.*

While not directly verifiable, they accord with experience to such a degree that departures cannot be detected, and the positions of the planets and their satellites are in such close agreement with their predicted places that this agreement is of the nature of a demonstration.

106. The absolute axes.—In kinematics the choice of the absolute axes was arbitrary. The state of affairs in mechanics is different. The principles just spoken of are asserted true of the motion of a particle referred to a *particular* set of axes *invariably connected with the so-called fixed stars.* These I term the *absolute axes.* Referred to any other set the *principles* must be modified, and this necessary modification will be studied in a subsequent chapter.

It will then be assumed, *and this is fundamental in the theory*, that unless otherwise stated the *absolute axes* are under consideration.

Isolated particle.—If we conceive all the matter of the universe except one particle to cease to exist, this particle is termed *isolated.* Of course no particle ever is *isolated,* but we may assert a statement true of an isolated particle, meaning thereby that it describes what would happen were we under these ideal conditions. A set of geometric points may be imagined as taking the place of the fixed stars and thus determining the absolute axes, they being a *geometric* and not a *physical* concept.

107. First principle of mechanics.—The first principle may be stated as follows:

An isolated particle has no acceleration with respect to the absolute axes.

This principle is also termed the *principle of inertia.*

108. Meaning of the first principle.—Since an isolated particle has no acceleration its velocity is constant and hence it either remains at rest or describes a straight line with constant speed; that is, its motion is rectilinear and uniform.

In other words, an isolated particle *persists* in its state of motion, and this property is termed *inertia.*

REMARK.—In virtue of this principle, a particle accelerated with respect to the absolute axes, that is, a particle possessing an *absolute acceleration,* must be in the presence of other particles.

It is to be observed that no assertion is made concerning a particle in the presence of other particles—it may or may not be accelerated. What is asserted is this: *If the particle is accelerated, then it is in the presence of other particles.*

109. Field of force.—A portion of space within a closed surface—which may be entirely at infinity—is termed a field of force when a particle abandoned *at rest* at any point in it takes an accelerated motion. The space occupied by the solar system and bounded by a spherical surface through the nearest fixed star is an example of such a field.

There is good reason to believe that this field extends through the space occupied by the stars themselves, and we are accustomed to regard infinite space as a field of force.

110. Uniform field of force.—If the acceleration which the particle takes is independent of the point where it is abandoned, and is the same for all points of the field *at the same instant,* the field is termed uniform.

111. Constant field of force.—If in a uniform field of force the acceleration is independent of the time the field is termed constant.

112. Superposition of fields of force.—By the first principle, wherever there is a field of force there must be matter. Suppose the matter present to consist of n particles anyhow distributed. Let $n=p+q$. Then p of those particles imagined as present alone would produce a field Φ_1, and the remaining q imagined as present alone would produce a field Φ_2. If all n are present we may imagine the two fields Φ_1 and Φ_2 as *supposed* producing a *resultant* field Φ, of which Φ_1 and Φ_2 are termed components. The field Φ_1 alone would produce a certain acceleration J_1 in a particle M placed in it. Similarly Φ_2 alone would produce a, in general different, acceleration J_2 on the same particle at the same point. The resultant field Φ produces an acceleration J on this particle. How are these three accelerations related?

113. Second principle of mechanics.—The question is answered by the following principle, known as the *second principle of mechanics:*

The acceleration which a particle takes in a resultant field of force is the geometric sum of the accelerations produced by the component fields, and is independent of the particle and of its motion.

114. Meaning of the second principle.—From this principle we draw the following conclusions:

1° The acceleration at any point in a field of force is the same whether the particle be abandoned there at rest or pass the point with a velocity different from zero.

2° The acceleration at any point in the field is the same for all particles.

3° If J be the acceleration produced by the resultant field Φ and J_1 and J_2 the accelerations produced by the component fields Φ_1 and Φ_2, these three accelerations satisfy the equipollence

$$\overline{J}=\overline{J_1}+\overline{J_2} \qquad [120]$$

4° Since the same reasoning can be applied to any number of component fields $\Phi_1 \dots \Phi_n$, we have in general

$$\overline{J}=\overline{J_1}+\dots+\overline{J_n} \qquad [121]$$

115. Reaction of the particle on the field.—Conceive a particle M in a field Φ produced by n other particles $M_1 \dots M_n$. Now the particle M together with any $(n-1)$ of the n particles may be looked upon as producing a field which acts on the remaining one. That is, conceive M_1 (say) in the field produced by M, $M_2 \dots M_n$. This particle M_1 takes a certain acceleration J_1 at the same instant that M, looked upon as acted on by the field of $M_1 \dots M_n$, takes the acceleration J. Is there any relation between J and J_1?

This question is answered by the following principle.

116. Third principle of mechanics.—This principle may be stated as follows:

Two isolated particles under their mutual actions take accelerations in opposite directions along the line joining them, and these accelerations are in a constant ratio.

117. Meaning of the third principle.—From this principle we conclude that two particles M_i and M_j take accelerations J_{ij} and J_{ji} which satisfy the equipollence

$$\overline{J_{ij}} + \rho_{ji}\overline{J_{ji}} = 0 \qquad [122]$$

and the algebraic equation

$$J_{ij} = \rho_{ji} J_{ji} \qquad [123]$$

where ρ_{ji} is a constant depending on the two particles.

118. Properties of the coefficients ρ_{ij}.—Consider any three particles M_i, M_j, and M_k. In the field of M_i the two particles M_j and M_k take accelerations J_{ji} and J_{ki} which satisfy

$$J_{ij} = \rho_{ji} J_{ji}$$

$$J_{ik} = \rho_{ki} J_{ki}$$

Whence

$$\frac{J_{ij}}{J_{ik}} = \frac{\rho_{ji}}{\rho_{ki}}$$

since by the second principle

$$J_{ji} = J_{ki}$$

Similarly, considering the fields produced by M_j and M_k, we have

$$\frac{J_{jk}}{J_{ji}} = \frac{\rho_{kj}}{\rho_{ki}}$$

and

$$\frac{J_{ki}}{J_{kj}} = \frac{\rho_{ik}}{\rho_{jk}}$$

Multiplying these equations we have

$$\frac{J_{ij}}{J_{ik}} \cdot \frac{J_{jk}}{J_{ji}} \cdot \frac{J_{ki}}{J_{kj}} = \frac{\rho_{ji}}{\rho_{ki}} \cdot \frac{\rho_{kj}}{\rho_{ij}} \cdot \frac{\rho_{ik}}{\rho_{jk}}$$

and making use of the relations

$$\begin{aligned} J_{ij} &= \rho_{ji} J_{ji} \\ J_{jk} &= \rho_{kj} J_{kj} \\ J_{ki} &= \rho_{ik} J_{ik} \end{aligned}$$

we obtain

$$1 = \frac{1}{\rho_{ki} \rho_{ij} \rho_{jk}}$$

Whence $$\rho_{jk} = \frac{\rho_{ji}}{\rho_{ki}} = \frac{\rho_{ik}}{\rho_{ij}} \qquad [124]$$

119. **Definition of mass.**—Writing the coefficients ρ_{ji} in the form

$$\rho_{ji} = \frac{m_j}{m_i}$$

the equipollence

$$\overline{J_{ij}} + \rho_{ji} \overline{J_{ji}} = 0$$

becomes

$$m_i \overline{J_{ij}} + m_j J_{ji} = 0$$

or $$m_i J_{ij} = m_j J_{ji} \qquad [125]$$

and relation [124] becomes

$$\frac{m_j}{m_k} = \frac{m_i}{m_k} \Big/ \frac{m_i}{m_j} = \frac{m_i}{m_k} \cdot \frac{m_j}{m_i}$$

HENCE, *the coefficients* $m_i, m_j, m_k \ldots m_r$ *are absolute constants depending only on the particles themselves and not on their accelerations.*

These constants are termed the *masses* of the particles, and since only their ratios are determined, one of them can be arbitrarily assigned, and the rest then become known.

120. Force.—The word *force* does not enter into the principles of mechanics, as they have been stated, and the problems of mechanics can be solved, in so far as they are capable of solution, without ever making use of the term.

But it is often convenient to employ the term in abbreviating statements, and it is introduced as follows:

When a particle M of mass m is accelerated by being in the presence of other particles it is said to be subjected to a *force F defined* by the equipollence (fig. 33)

$$\bar{F} = m\bar{J} \qquad [126]$$

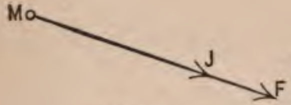


FIG. 33.

The force acting on a particle is, then, a *vector* whose direction is that of the acceleration and whose magnitude is the product of the mass by the acceleration.

121. Observations on the notion of force.—Force is thus seen to be a purely mathematical and not a physical concept. It has no objective existence, but is merely a name for a certain product in mechanics. To say that a force acts on a particle is simply to assert that the particle is accelerated, and this acceleration is all that is ever observable. We may produce the sensation of *effort* by attempting to move bodies, but in every case the *force* which we exert is only a component of the acting force, and inasmuch as a particle at rest may be conceived as having accelerations whose geometric sum is zero, so it may be conceived as acted on by forces one of which is exerted by ourselves.

122. Composition of forces. Resultant.—The second principle gives at once the rule for the composition of forces. For if a system of forces

$$F_1 \ldots F_n$$

act on a particle M of mass m, producing accelerations

$$J_1 \ldots J_n$$

then, by the second principle, the resultant acceleration is

$$\overline{J} = \overline{J_1} + \ldots + \overline{J_n}$$

and the resultant force is

$$\overline{F} = m\overline{J} = m(\overline{J_1} + \ldots + \overline{J_n})$$

or

$$\overline{F} = \overline{F_1} + \ldots + \overline{F_n} \qquad [127]$$

HENCE, *the resultant of a system of forces acting on a particle is their geometric sum.*

From this follows that:

1° The projection on a plane of the resultant of a system of forces acting on a particle coincides with the resultant of the projections of the forces on that plane.

2° The projection on an axis of the resultant of a system of forces acting on a particle coincides with the algebraic sum of the projections of the forces on that axis.

3° The resultant of two forces is the diagonal of the parallelogram constructed on them. This is termed *their parallelogram.*

4° The resultant of three forces is the diagonal of the parallelopiped constructed on them. This is termed *their parallelopiped.*

5° The resultant of a system of forces is equipollent to the vector which closes any one of the polygons formed by placing them end to end. This polygon is termed the *polygon of the forces.*

6° In particular, the polygon of two forces becomes a triangle and is termed *their triangle.*

123. Decomposition of forces. Components.—To *decompose* a force is to replace it by a system of forces whose resultant is equipollent to the given force. The forces of the system are termed *components* of the given force.

In particular a force may be decomposed in a single manner into:

1° Two forces in two directions arbitrarily chosen in any plane through the force itself.

For in any plane through the force, one and but one parallelogram can be constructed having its sides in assigned directions and having a given diagonal.

2° Two others, one of which is chosen at will.

For, in the plane defined by the given force and the arbitrarily chosen force, one and but one parallelogram can be constructed having an assigned side and diagonal.

3° Two others, of which one has an arbitrary direction and the other is in a plane of arbitrary orientation. In particular, into two others of which one has an arbitrary direction and the other is in a plane perpendicular to this direction.

For, in the plane defined by the arbitrarily chosen direction and the given force, the second direction is determined by its intersection with the arbitrarily chosen plane, and the problem is reduced to the first case.

In the particular case where the arbitrarily chosen plane is perpendicular to the chosen direction, the components are the projections of the given force on the chosen direction and the normal plane.

4° Three others along directions not in the same plane but otherwise arbitrary.

For one, and but one parallelopiped can be constructed having an assigned diagonal and assigned directions for its edges.

In particular, the components may be taken parallel to three coordinate axes, and are then equipollent to the projections of the given forces on these axes.

124. **Equilibrium of a free particle.**—When the resultant acceleration which a system of forces imposes on a particle is zero, the particle is said to be in *equilibrium*.

The system of forces is also said to be in *equilibrium*.

Since the resultant force and the resultant acceleration are connected by the equipollence

$$\overline{F} = m\overline{J}$$

it follows that if J is zero F is also zero.

HENCE, *the necessary and sufficient condition that a system of forces acting on a particle be in equilibrium is that their resultant shall be zero.*

According as one wishes to study the conditions of equilibrium by geometric or analytic methods, one employs the polygon of forces or their projections on the coordinate axes.

COROLLARIES.—1° In order that two forces acting on a particle be in equilibrium it is necessary and sufficient that they be equal and opposite.

2° In order that three forces acting on a particle be in equilibrium it is necessary and sufficient that they be equipollent to the three sides of a triangle taken in the same order.

3° If a system of forces acting on a particle is in equilibrium, each of them is equal and opposite to the resultant of the others.

4° If a system of forces is in equilibrium, the system formed by their projections on any plane or axis is in equilibrium.

5° If a system of forces in equilibrium act on a particle in motion, its motion is uniform rectilinear, as if these forces did not exist.

6° More generally, whatever be the forces acting on a free particle, a system of forces in equilibrium may be introduced or suppressed. For every system of forces may be replaced by its resultant, and this is not changed by the addition or subtraction of forces whose geometric sum is zero.

125. Force acting on a particle at rest.

THEOREM.—*Whenever the velocity of a particle is zero, and therefore whenever the direction of motion changes, and in particular when the particle starts from rest, the force acting is tangent to the path.*

For
$$\overline{J}=\frac{D\overline{V}}{Dt}=\lim \left.\frac{\overline{V_1}-\overline{V}}{\Delta t}\right]_{\Delta t=0}$$

and if $V=0$, then the direction of this limit is the direction toward which V_1 tends and is therefore tangent to the path.

Hence $\overline{F}$ $(=m\overline{J})$ is also tangent to the path at this point.

126. New definition of direction of force.—We may then define the direction of the force as the direction of the *first* *element of the path* which the particle describes *starting from rest.*

127. Tangential and normal force.—In general the force is not tangent to the path, but it always lies in the osculating plane, since the acceleration is in this plane. It may therefore be decomposed into two components, one along the tangent and the other along the principal normal.

Projecting the equipollence

$$\overline{F}=m\overline{J}$$

on the tangent, we have

$$F_s=mJ_s=m\frac{d^2s}{dt^2} \qquad [128]$$

and on the normal

$$F_n=mJ_n=\frac{m}{\rho}\left(\frac{ds}{dt}\right)^2 \qquad [129]$$

128. Resistance.—When a force produces a negative acceleration it is termed a *resistance,* and this is the case when the direction of the acceleration makes an obtuse angle with the direction of the velocity.

CHAPTER X

THE UNITS OF MECHANICS

129. The units of kinematics.—130. The units of mechanics.—131. The three fundamental units. Derived units.—132. Systems of units.—133. C.G.S. System.—134. M.K.S. System.—135. English System.—136. Units of astronomy.—137. Units of force.—138. Remark.

129. The units of kinematics.—In geometry a single unit is sufficient; that of length and any one of the quantities, centimeter, meter, kilometer, inch, foot, yard, mile, etc., can equally well be used. In kinematics a new element was introduced, namely, *time*, and the unit adopted was the *mean solar second.* It is understood that when more convenient any multiple of this, as a minute, hour, day, year, etc., may be used.

130. The units of mechanics. -In mechanics it was necessary to introduce a second new quantity, namely, *mass.* The *gram* or any multiple or submultiple may be taken as the unit of mass.

131. The three fundamental units. Derived units.—As we shall see, these three units of length, time, and mass being defined, all the rest are determined as they are expressible in terms of these. These three are accordingly termed *fundamental units* and the rest are known as *derived units.*

132. Systems of units.—Various systems of units may be built up depending on the choice of the fundamental units. Three are in common use, as follows:

133. C.G.S. System.—If the centimeter, gram, and second are taken as fundamental units, the system is termed the *Centimeter-Gram-Second System.*

134. M.K.S. System.—If the meter, kilogram, and second are taken as fundamental units, the system is termed the *Meter-Kilogram-Second System.*

135. English System.—If the foot, pound, and second are taken as fundamental units, the system is termed the *Foot-Pound-Second* or *English System.*

136. Units of astronomy.—Because of the large magnitudes there dealt with it has been found convenient in astronomy to adopt as the unit of distance the *mean distance* from the Earth to the Sun, and as the unit of mass the mass of the Earth or Sun. The unit of time is the second day or year. These are the *astronomical units.*

137. Units of force.—The unit of force being a derived unit depends on the fundamental units, and hence is different in the three systems described above.

From

$$F=mJ$$

we see that $F=1$ when $m=1$ and $J=1$.

HENCE, *the unit of force is that force which acting on unit mass for unit time gives it unit acceleration.*

In the C.G.S. system it is called the *dyne,* in the M.K.S. system it has no name, but is referred to as the M.K.S. unit of force, while in the English system it is known as the *poundal.*

The remaining derived units will be defined as occasion arises.

138. Remark.—It is to be carefully observed that just as soon as the unit of length is chosen those of area and volume are fixed, and as soon as the units of length and time are chosen those of velocity and acceleration are fixed; similarly, as soon as the units of length, time, and mass are chosen all the derived units of mechanics are determined, although not all of them have names.

CHAPTER XI

GENERAL NOTIONS ON THE FORCES OF NATURE

139. Principle of the equality of action and reaction.—140. Analytic expression of the common force between two free particles.—141. Central forces.—142. Universal gravitation.—143. Note.

139. Principle of the equality of action and reaction.—Newton enunciated the following principle under the name of the *Principle of the Equality of Action and Reaction.*

If a particle M is acted on by a force F due to a second particle M′, then this force is directed along the line MM′ and M′ is acted on by a force F′ equal and opposite to F.

This is nothing more than a *restatement of the third principle in terms of force,* for if M of mass m is acted on by a force F, then it takes an acceleration J defined by

$$\overline{J}=\frac{\overline{F}}{m},$$

and by the third principle M' takes an acceleration J' defined by

$$\overline{J}'=-\rho.\overline{J}=-\frac{m}{m'}J=-\frac{F}{m}$$

and hence is acted on by a force F' defined by

$$\overline{F'}=m'\overline{J'}=-\overline{F}$$

The forces F and F' are then equal in magnitude, and are either directed toward each other, as in fig. 34_1, or away from each other, as in fig. 34_2.

One of these forces, either F or F', is termed *the action*, and the other the *reaction*.

By the second principle of mechanics this principle of the equality of action and reaction is immediately extended to the mutual actions of two systems of particles (S) and (S').

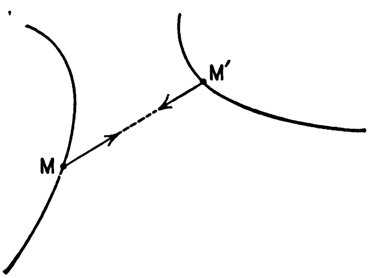


Fig. 34_1.

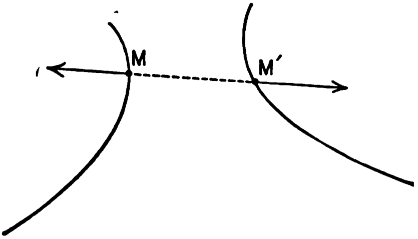


Fig. 34_2.

If the particles of a system (S) *exert certain forces on the particles of a system* (S'), *then the particles of* (S') *exert on the particles of* (S) *equal and opposite forces.*

140. **Analytic expression of the common force between two free particles.**—Denoting the two particles by M and M', and their masses, velocities, and accelerations by m, V, J and m', V', J', we have

$$F = mJ = m'J'$$

Now by the second principle J does not depend on m or V, and J' does not depend on m' or V'. Hence J can be a function of m', V', and r only, where r is the distance between M and M'. Similarly JJ' can be a function of m,

V, and r only. Again, since M and M' may be interchanged without affecting J and J' in magnitude, the law of dependence must be the same in the two cases.

We may therefore write

$$J=\phi(m', V', r) \qquad J'=\phi(m, V; r)$$

Whence $$m\cdot\phi(m', V', r)=m'\cdot\phi(m, V, r)$$

Therefore $$\frac{\phi(m', V', r)}{m'}=\frac{\phi(m, V, r)}{m}$$

Now, since m, m', V, V' are all independent the two members of this equation can depend on r only, and we may write

$$\frac{\phi(m', V', r)}{m'}=\frac{\phi(m, V, r)}{m}=\Phi(r)$$

Whence $$F=mJ=m'J'=mm'\Phi(r) \qquad [130]$$

141. **Central forces.**—A force of the type

$$F=m\cdot m'\cdot\Phi(r)$$

is termed a *central force*, and

HENCE, *All forces between free particles are central forces.*

142. **Universal gravitation.**—Newton, interpreting Kepler's laws of the motions of the planets, showed that all the motions of the celestial bodies are determined with a remarkable precision if we assume that:

The central force between two free particles is one of attraction and varies inversely as the square of their distance apart.

In this case

$$\Phi(r)=\frac{G}{r^2}$$

where G is a constant for any two particles. A discussion of Kepler's laws also shows that *G is the same for every pair of particles.* In the C.G.S. system of units

$$G=\frac{666}{10^6} \text{ (nearly)}$$

The central force of attraction of any two free particles may then be written

$$F=G\frac{m\cdot m'}{r^2} \qquad [131]$$

and for this reason central forces of this type are often called *Newtonian forces.*

143. NOTE.—I do not here include electric, magnetic, and electromagnetic forces between free particles, for the reason that these belong rather to theoretical physics than to rational mechanics. Forces of capillarity and cohesion are not included for the reason that the particles are not there free.

As to the force between two free particles when the distance between them is small, that is, comparable to molecular distances, and which does not seem to be Newtonian, special monographs should be consulted.

CHAPTER XII

DETERMINATION OF THE LAW OF FORCE FROM THE MOTION PRODUCED

144. Finite and differential equations of motion.—145. The two general problems of the mechanics of a particle in terms of force.—146. The first problem.—147. The different forces capable of producing a given rectilinear motion.—148. The force producing uniformly accelerated motion.—149. Law of force producing harmonic motion.—150. Law of force producing harmonic motion with coefficient of extinction.—151. On the resistance producing extinction.—152. Law of force producing uniform curvilinear motion.

144. Finite and differential equations of motion.—The equations which define a motion with respect to three coordinate axes are of the form

$$x=f(t) \qquad y=f_1(t) \qquad z=f_2(t)$$

where f, f_1, and f_2 are uniform and continuous functions of t. It is only when these three functions are known that a motion is completely determined.

The projections of the velocity are given by

$$V_x=f'(t) \qquad V_y=f_1'(t) \qquad V_z=f_2'(t)$$

and the projections of the acceleration by

$$J_x=f''(t) \qquad J_y=f_1''(t) \qquad J_z=f_2''(t)$$

Projecting the equipollence

$$\bar{F} = m\bar{J}$$

on the three axes, we have for the projections of the force

$$F_x = mJ_x \qquad F_y = mJ_y \qquad F_z = mJ_z$$

These equations, which are usually written

$$mx'' = F_x \qquad my'' = F_y \qquad mz'' = F_z \qquad [132]$$

are called the *differential equations of the motion.*

145. The two general problems of the mechanics of a particle in terms of force.—In Art. 103 the general problem of mechanics and its converse were stated in terms of acceleration, and it was seen that if we know the initial displacement and velocity of the particle and the acceleration *as a function of the time,* the motion is determined. The two problems may now be stated as follows:

1° *Given, by observation or otherwise, the motion—that is, the equations of motion—of a particle, to determine the law of force.*

2° *Given the laws which the forces acting obey, to determine the motion.*

The second problem is by far the more important, and also the more difficult. However, the first occurs frequently enough to make its study worth while. It will be seen that the solution of the first depends on the operations of the *differential* calculus, which can always be effected, while the second leads to the operations of the *integral* calculus, which can be effected only in a limited number of cases.

146. The first problem.—Suppose the motion given by its finite equations

$$x=f(t) \qquad y=f_1(t) \qquad z=f_2(t) \qquad [132]$$

The projections of the velocity are immediately

$$V_x=f'(t) \qquad V_y=f_1'(t) \qquad V_z=f_2'(t) \qquad [133]$$

and of the force

$$F_x=mf''(t) \qquad F_y=mf_1''(t) \qquad F_z=mf_2''(t) \qquad [134]$$

The force is then determined in direction and magnitude and the problem *seems* solved. In reality it is not solved if we are looking for a law of force occurring in nature, for in the components of the force as determined above the time enters explicitly, which it never does in a natural law, inasmuch as a particle under the *same conditions*—that is, in the *same relative position* with respect to other particles—always takes the same motion independent of the time.

This principle is termed the *Principle of the Uniformity of Nature.*

It is then in terms of the position and velocity of the particle that the force must be expressed. This may be done by eliminating t between equations [132] and [133] giving one law of force independent of the velocity, or between [133] and [134] giving a second law depending on the velocity.

Which, then, is the *natural* law which produces the motion?

If it is known from physical considerations that the force does not depend on the velocity, as in the case of free particles, then the first law is the one sought. Otherwise the problem is indeterminate unless the choice can be made from physical conditions. Usually it can.

147. The different forces capable of producing a given rectilinear motion.—If the motion is rectilinear, then the acceleration and hence the force is directed along the path.

Let
$$x=f(t)$$

be the equation of motion, then the velocity is

$$x'=f'(t)$$

and the acceleration is

$$x''=f''(t)$$

The force is therefore

$$F=mf''(t)$$

Eliminating (t) between the first two equations, we have the law of force

$$F=m\phi(x) \qquad [135]$$

This expression can represent a law of nature, and if we know that the force does not depend on the velocity, it represents the *one* law capable of producing the motion.

But in the absence of any information to the contrary we may also eliminate t between the second pair and get the law of force

$$F=m\psi(x') \qquad [136]$$

which can also represent a natural law.

Allowing x and x' both to enter, we can get an infinite number of laws of force.

148. The force producing uniformly accelerated motion.—Here there is no ambiguity. The acceleration is constant and there is one type of force only capable of producing the motion, namely, a constant force.

149. Law of force producing harmonic motion.—All harmonic vibrations of the same period are represented by the equation

$$x=a\sin(nt-\phi)$$

Whence
$$x'=an\cos(nt-\phi)$$
$$x''=-n^2[a\sin(nt-\phi)]$$

Hence there is one solution only:

$$F=mx''=-mn^2x \qquad [137]$$

Therefore, *there is but one type of force capable of producing all harmonic vibrations of same period in a particle. It is an attractive central force varying directly as the distance of the particle from the centre.*

150. Law of force producing harmonic motion with coefficient of extinction.—Consider the rectilinear motion whose equation is

$$x=ae^{-\alpha t}\sin(nt-\phi)$$

where e is the base of the Naperian system of logarithms and α is a number, generally small and essentially positive, which is called the *coefficient of extinction.*

For $\alpha=0$ we have the harmonic motion, but for $\alpha>0$ and $t=\infty$ we have $x=0$, which shows that the motion dies down.

Increasing t by $\frac{2k\pi}{n}$, where k is a positive integer, the sine does not change, but the displacement x is multiplied by

$$e^{-\frac{2k\pi\alpha}{n}}$$

and for a sufficiently large number of periods becomes as small as we please.

To determine the law of force capable of producing this motion we must calculate the acceleration.

From the equation of motion

$$x' = ae^{-\alpha t}[-\alpha \sin (nt-\phi) + n \cos (nt-\phi)]$$

and

$$x'' = ae^{-\alpha t}[-(n^2-\alpha^2) \sin (nt-\phi) - 2\alpha n \cos (nt-\phi)]$$

Combining the first of these with the equation of motion,

$$ae^{-\alpha t} \sin (nt-\phi) = x$$

there results

$$nae^{-\alpha t} \cos (nt-\phi) = x' + \alpha x$$

Whence $$x'' = -(n^2-\alpha^2)x - 2\alpha(x'+\alpha x)$$

and $$F = mx'' = -m(n^2+\alpha^2)x - 2m\alpha x'$$

151. On the resistance producing extinction.—If $\alpha = 0$, we have

$$F = -mn^2x$$

the law of force producing harmonic motion.

If $\alpha \mp 0$, then F may be regarded as the algebraic sum of the two components

$$F_1 = -m(n^2 + \alpha^2)x$$

which is directed toward the centre and is proportional to the distance of the particle from the centre, and

$$F_2 = -2m\alpha x'$$

of an entirely different nature. Instead of depending on the displacement of the particle it depends on its velocity and is always directed opposite to the velocity. It is, then, always a resistance. It vanishes when the velocity is zero and hence cannot produce motion, but has the effect of slowing down the motion produced by F_1 and finally brings the particle to rest at the origin.

152. Law of force producing uniform curvilinear motion.—We have seen that in any curvilinear motion the acceleration lies in the osculating plane, and its projections on the tangent and principal normal are

$$J_t = \frac{d^2s}{dt^2}$$

$$J_n = \frac{V^2}{\rho} = \frac{\left(\frac{ds}{dt}\right)^2}{\rho}$$

where s is the arc described measured from some initial point and ρ is the radius of curvature.

If the motion is uniform

$$V=\frac{ds}{dt}=V_0, \text{ a constant}$$

and $$J_t=\frac{d^2s}{dt^2}=0$$

HENCE, *in uniform curvilinear motion the acceleration is directed along the normal and has the value*

$$J=\frac{V_0^2}{\rho}$$

The force acting is then

$$F=mJ=\frac{mV_0^2}{\rho} \qquad [138]$$

directed along the principal normal, and is therefore perpendicular to the path.

If the path is a circle, $\rho=R$, the radius, and

$$F=\frac{mV_0^2}{R}$$

CHAPTER XIII

MOTION PRODUCED BY FORCES OBEYING KNOWN LAWS

153. Statement of the problem.—**154.** Attraction toward a fixed centre proportional to the distance.

153. Statement of the problem.—In the problem of the preceding chapter we were given the finite equations of motion

$$x=f(t) \qquad y=f_1(t) \qquad z=f_2(t) \qquad [139]$$

and were required to find the law of force, *independent of the time*, from the projections

$$F_x=mf''(t) \qquad F_y=mf_1''(t) \qquad F_z=mf_2''(t)$$

For this it was sufficient to take the first and second derivatives of f, f_1, and f_2 with respect to the time,—a simple and determinate problem,—and then eliminate t between these equations. We saw how this could be done in some simple cases.

Here, on the contrary, we are given the projections of the force acting in terms of the *coordinates, velocities, and the time.*

$$F_x(x,y,z,x',y',z',t) \qquad F_y(x,y,z,x',y',z',t) \qquad F_z(x,y,z,x',y',z',t)$$

and are required to pass back to the finite equations of motion of the particle.

The differential equations of the motion are at once obtained by writing

$$mx'' = F_x \qquad my'' = F_y \qquad mz'' = F_z \qquad [140]$$

These equations [139] and [140] which define the *same motion* are equivalent, but the process of passing back from [140] to [139] is by no means simple, and indeed not always possible.

As to the force F it may or may not depend explicitly on the velocities and the time. In the case of a free particle it depends only on the coordinates—that is, on the position of the particle. In the case of a particle moving in a resisting medium it will in general depend on the velocities also, and in certain problems of constrained motion it depends explicitly on the time.

Under these conditions the process of passing from [140] to [139] is one of the most difficult operations of the integral calculus. Moreover, whenever we pass from a derivative back to the primitive function an *arbitrary* constant is introduced. In this case, therefore, since there are three unknown functions of the time, namely, x, y, and z, all entering to the second derivative, there will, in general, be introduced into the solution *six arbitrary constants.* The problem is therefore *very indeterminate.* The theory of differential equations, however, tells us that, given the position and velocity of the particle at *any one instant,* there *exists* a solution, and a *single* one. The effective determination of this solution is a matter quite apart from its existence.

The problem should then be stated as follows:

Being given:

1° *The law of the force or forces acting on the particle,*

2° *The initial position and velocity of the particle,*

determine its motion, that is, its position at each instant.

154. Attraction toward a fixed centre proportional to the distance.—A special problem will illustrate the difficulties and also the method of procedure.

Let (fig. 35) O be a fixed point and M a particle of mass m acted on by a force F always directed toward O and proportional to MO.

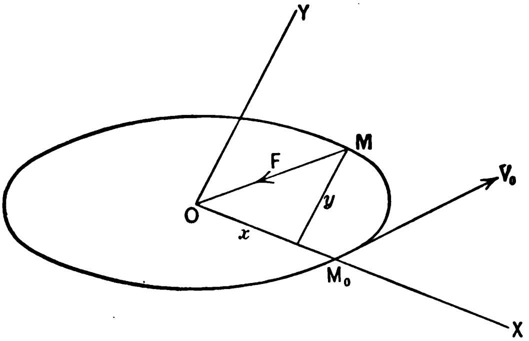


Fig. 35.

Let M_0 be the initial position of M and V_0 its initial velocity. Take the X-axis through M_0 and the Y-axis in the plane of OX and V_0, the Z-axis being then perpendicular to this plane.

Let x, y, z be the coordinates of M at the instant t,

then $$mx'' = F_x \qquad my'' = F_y \qquad mz'' = F_z$$

Denoting by n^2 a constant, we may write

$$F = m \cdot n^2 \cdot r$$

where r denotes the distance OM.

Then $\quad F_x = -F \cdot \frac{x}{r} \qquad F_y = -F \cdot \frac{y}{r} \qquad F_z = -F \cdot \frac{z}{r}$

and hence

$$\left.\begin{aligned} F_x &= -m \cdot n^2 \cdot x \\ F_y &= -m \cdot n^2 \cdot y \\ F_z &= -m \cdot n^2 \cdot z \end{aligned}\right\}$$

The *differential* equations of motion are then

$$x'' = -n^2 x \qquad y'' = -n^2 y \qquad z'' = -n^2 z \qquad [141]$$

The first of these contains x only, the second y only, and the third z only, and may therefore be studied separately. This is, in general, *not* the case. If, for example, F, instead of varying as r, varied inversely as r^2, we should have

$$F = \frac{m \cdot n^2}{r^2},$$

and $\quad F_x = -\frac{m \cdot n^2 \cdot x}{r^3}$

In this case each differential equation of motion, instead of containing one coordinate only, would contain the distance

$$r = \sqrt{x^2 + y^2 + z^2}$$

to the $-^3/_2$ power. These equations could not then be treated separately, but would form a system of *simultane-*

ous differential equations in three unknowns, and the difficulty of finding a solution would be immensely increased.

To return to equations [141], the equation

$$x'' = -n^2 x \qquad [142]$$

expresses that in the projection of the motion on the X-axis the acceleration is always directed toward O and is proportional to the abscissa of the projection of M on the X-axis. But we have seen that every harmonic motion along OX with O as centre and of period $\frac{2\pi}{n}$ has exactly this equation, no matter what the amplitude a and the phase ϕ may be. Hence, *the finite equation of motion*

$$x = a \sin (nt - \phi)$$

is a solution of

$$x'' = -n^2 x$$

no matter what values the constants a and ϕ may have.

Putting $-a \sin \phi \equiv A$, $a \cos \phi \equiv B$, we may write

$$x = A \cos nt + B \sin nt \qquad [143]$$

In fact we have

$$x' = n[-A \sin nt + B \cos nt]$$

$$x'' = -n^2[A \cos nt + B \sin nt] = -n^2 x$$

which shows that [143] satisfies [142]. In the same way

$$y = A_1 \cos nt + B_1 \sin nt \qquad [144]$$

$$z = A_2 \cos nt + B_2 \sin nt \qquad [145]$$

where A_1, B_1, A_2, B_2 are *arbitrary* constants. These are then the finite equations of motion, containing six arbitrary constants, which must be so determined as to satisfy the initial conditions

$$\left.\begin{array}{lll} x=OM_0=x_0 & y=0 & z=0 \\ x'=V_{0x} & y'=V_{0y} & z'=V_{0z} \end{array}\right\}$$

for $t=0$.

Since z and z' vanish for $t=0$, it is easily shown that z is identically zero, and the motion is in the plane of XY.

In fact, from [145]

$$z'=n[-A_2 \sin nt+B_2 \cos nt]$$

whence, for $t=0$

$$z=A_2=0 \qquad z'=-nB_2=0$$

and therefore $A_2=B_2=0$.

We have, then, only to consider the equations in x and y. For $t=0$ we have

$$x=A\ =x_0 \qquad x'=nB\ =V_{0x}$$

$$y=A_1=\ 0 \qquad y'=nB_1=V_{0y}$$

hence the equations of motion are

$$\left.\begin{array}{l} x=x_0 \cos nt+\dfrac{V_{0x}}{n} \sin nt \\ y=\dfrac{V_{0y}}{n} \sin nt \end{array}\right\} \qquad [146]$$

Two cases present themselves according as the initial velocity V_0 has or has not the direction of OX.

In the first case

$$V_{0x}=V_0 \qquad V_{0y}=0$$

and the equations of motion are

$$\left.\begin{aligned} x&=x_0 \cos nt+\frac{V_0}{n} \sin nt \\ y&=0 \end{aligned}\right\}$$

the velocity V_0 being positive or negative according as it is directed away from or toward O.

The motion is rectilinear and harmonic, and the amplitude and phase are easily deduced.

In the second case let V_0 make an angle α with the positive direction of the X axis,

then $$V_{0x}=V_0 \cos \alpha \qquad V_{0y}=V_0 \sin \alpha$$

whence $$B=\frac{V_0 \cos \alpha}{n} \qquad B_1=\frac{V_0 \sin \alpha}{n}$$

and $$\left.\begin{aligned} x&=x_0 \cos nt+\frac{V_0 \cos \alpha}{n} \sin nt \\ y&=\frac{V_0 \sin \alpha}{n} \sin nt \end{aligned}\right\}$$

The equation of the path obtained by eliminating t between these two equations is

$$(x \sin \alpha-y \cos \alpha)^2+\frac{x_0^2 n^2}{V_0^2} y^2=x_0^2 \sin^2\alpha$$

which is easily seen to be an ellipse whose diameter conjugate to OX is parallel to V_0.

If $\alpha=90°$, and hence V_0 perpendicular to OX, the the ellipse

$$\frac{x^2}{x_0{}^2}+\frac{n^2}{V_0{}^2}y^2=1$$

Observe that from equations [146]

$$\left.\begin{array}{l} x'=n\left(-x_0 \sin nt+\dfrac{V_0}{n} \cos nt\right) \\ y'=V_{0y} \cos nt \end{array}\right\}$$

whence

$$\frac{x'^2}{n^2}+x^2=\frac{V_{0x}{}^2}{n^2}+x_0{}^2=\text{constant}$$

Similarly

$$\frac{y'^2}{n^2}+y^2=\frac{V_{0y}{}^2}{n^2}=\text{constant}$$

Adding, and observing that

$$x'^2+y'^2=V^2$$

$$x^2+y^2=r^2$$

we have

$$V^2+n^2r^2=V_0{}^2+n^2x_0{}^2=\text{constant}$$

which shows that the velocity V of the particl only on its distance r from O, the *centre of attra*

This is a particular case of a more gen known as the *theorem of Vis Viva.*

From the first two of equations [141]

$$x'' = -n^2 x$$
$$y'' = -n^2 y$$

we see that

$$yx'' - xy'' = 0$$

Hence

$$\text{Areal } J = 0$$

and therefore

$$\text{Areal } V = \text{constant}$$

The elliptic path is then described under the law of constant areal velocity.

To resume:

1° In the most general case, that is, whatever the initial conditions, an attraction toward a fixed centre proportional to the distance produces a periodic elliptic motion, whose period is $T = \frac{2\pi}{n}$, where n^2 is the coefficient of attraction.

The ellipse has its centre at the centre of attraction, and admits the directions of the initial radius vector and velocity as conjugate directions; the semiconjugate diameters are equal to the initial radius vector x_0 and the ratio $\frac{V_0}{n} = \frac{V_0 T}{2\pi}$ of the initial velocity V_0 of the particle to the angular velocity $n = \frac{2\pi}{T}$ corresponding to the period of the elliptic motion.

2° In the case where the initial velocity is perpendicular to the initial radius vector, these two conjugate diameters become the axes of the ellipse.

3° If, moreover, the relation

$$V_0 = nx_0$$

holds between the initial velocity V_0 and the initial radius vector x_0 and the angular velocity n—that is, if the initial velocity V_0 *corresponds* to the angular velocity n—the motion is circular and uniform with angular velocity n.

4° If the initial velocity coincides in direction with the initial radius vector, the motion reduces to harmonic rectilinear motion.

5° In every case the motion, either elliptic or circular, may be considered as the resultant of two harmonic motions of the same period along two directions, oblique or rectangular.

6° The velocity has the same magnitude each time the particle is the same distance from the origin.

7° The path is always described under the law of constant areal velocity.

REMARK.—This attractive force proportional to the distance arises whenever a system is slightly displaced from its position of equilibrium. We shall find an important application in the pendulum. Other applications are frequent in optics, acoustics, and electricity.

CHAPTER XIV

RELATIVE MOTION

155. Relative axes.—156. Relative force.—157. Problems of relative motion.—158. First problem.—159. Second problem.

155. Relative axes.—In the previous chapters we have referred the motion to the *absolutely fixed axes*. In certain classes of problems this is inconvenient. For instance, in considering the motion of the planets about the Sun, it is convenient to refer these motions to a set of axes having the origin at the centre of the Sun and remaining parallel to themselves. For the motion of particles on or near the Earth's surface it is convenient to refer to a set of axes fixed on the Earth, and in this case the origin is not only *not* fixed, but the directions of the axes change as the Earth moves in its orbit and about its axis. These are the *relative axes.*

156. Relative force.—I shall define the *relative force* as the product of the mass by the relative acceleration, that is, by the equipollence

$$\overline{F_r} = m \cdot \overline{J_r} \qquad [147]$$

157. Problems of relative motion.—The problems of relative motion may be stated in a manner analogous to those of absolute motion.

1° *Given the relative motion of a particle to determine the law of force.*

2° *Given the law of force and the initial conditions to determine the relative motion.*

These problems would play the same rôle and could be attacked in the same way as the corresponding problems in absolute motion if we had a set of *principles for relative motion.* It is *a priori* evident, however, that the absolute motion being determined by these principles, the relative is also, and it becomes simply a question of restating them for the relative axes.

158. First problem.—Such a restatement, while not very difficult, is neither necessary nor of advantage, for, in the case of the first problem, if the equations of motion be

$$\xi=\phi(t) \qquad \eta=\phi_1(t) \qquad \zeta=\phi_2(t)$$

the relative velocity and acceleration can be obtained by differentiation, and one or more *laws of relative force* are arrived at by eliminating t, and this entirely independent of the motion of the relative axes.

If it is desired to obtain the *law of absolute force* from the equations of relative motion, the motion of the relative axes must also be given. Suppose this done, that is, suppose o, o_1, o_2; a, b, c; a_1, b_1, c_1; a_2, b_2, c_2 to be given functions of the time, the absolute force is then

$$m\overline{J_a}=m(\overline{J_c}+\overline{J_r}+\overline{J_t}) \qquad [148]$$

where the projections of J_c, J_r and J_t are given in Art. 100. Eliminating t between this equipollence, the equipollence

$$\overline{V_a}=\overline{V_c}+\overline{V_r}$$

and the equations of motion

$$\xi = \phi(t) \qquad \eta = \phi_1(t) \qquad \zeta = \phi_2(t)$$

we have the various possible laws of absolute force expressed either in terms of the absolute or the relative coordinates and velocities, as we please.

159. Second problem.—Given the law of relative force and the initial relative conditions, the second problem is reduced to the integration of the differential equations of relative motion:

$$m\xi'' = F_{r_\xi} \qquad m\eta'' = F_{r_\eta} \qquad m\zeta'' = F_{r_\zeta} \qquad [149]$$

If, on the other hand, it is the law of absolute force which is given, and the relative motion is desired, then the relations of Art. 100 must be made use of.

From

$$\overline{J_a} = \overline{J_c} + \overline{J_r} + \overline{J_t}$$

we have at once

$$\overline{F_r} = m\overline{J_r} = m(\overline{J_a} - \overline{J_c} - \overline{J_t}) = m\left(\frac{\overline{F_a}}{m} - \overline{J_c} - \overline{J_t}\right) \qquad [150]$$

where J_a, J_c, and J_t are here to be expressed in terms of the *relative* coordinates and velocities by means of the relations

$$\left.\begin{aligned} x &= o + a\xi + b\eta + c\zeta \\ y &= o_1 + a_1\xi + b_1\eta + c_1\zeta \\ z &= o_2 + a_2\xi + b_2\eta + c_2\zeta \end{aligned}\right\}$$

$$\left.\begin{aligned} x' &= (o' + a'\xi + b'\eta + c'\zeta) + (a\xi' + b\eta' + c\zeta') \\ y' &= (o_1' + a_1'\xi + b_1'\eta + c_1'\zeta) + (a_1\xi' + b_1\eta' + c_1\zeta') \\ z' &= (o_2 + a_2'\xi + b_2'\eta + c_2'\zeta) + (a_2\xi' + b_2\eta' + c_2\zeta') \end{aligned}\right\}$$

The differential equations of relative motion then become

$$\left.\begin{aligned} m\xi'' &= F_{a_\xi} - mJ_{c_\xi} - mJ_{t_\xi} \\ m\eta'' &= F_{a_\eta} - mJ_{c_\eta} - mJ_{t_\eta} \\ m\zeta'' &= F_{a_\zeta} - mJ_{c_\zeta} - mJ_{t_\zeta} \end{aligned}\right\} \qquad [151]$$

and the problem is reduced to the integration of these equations. The right-hand members will in general contain the time explicitly, so that the equations [151] are of the *form*

$$\left.\begin{aligned} \zeta'' &= \Psi(\xi, \eta, \zeta, \xi', \eta', \zeta', t) \\ \eta'' &= \Psi_1(\xi, \eta, \zeta, \xi', \eta', \zeta', t) \\ \zeta'' &= \Psi_2(\xi, \eta, \zeta, \xi', \eta', \zeta', t) \end{aligned}\right\} \qquad [152]$$

CHAPTER XV

MOTION OF A FREE PARTICLE ON EARTH'S SURFACE

160. The relative axes.—161. Differential equations of motion.—162. First approximation.—163. Second approximation.

160. The relative axes.—Refer the motion to a set of axes having the origin at a point A near the Earth's surface and orientated in the following manner: Take the axis of ζ along the *direction of the geometric difference of the absolute and convective acceleration of* A, the axis of ξ in the plane of the meridian through A drawn toward the south; and the axis of η toward the east. Denote by g *the geometric difference of the absolute and convective accelerations of any point* $M(\xi, \eta, \zeta)$. The acceleration g will then depend, both in magnitude and direction, on the relative coordinates ξ, η, and ζ of M, but for a *first approximation may be taken as constant and equal to its value at* A, since for any region about A in which the motion of a particle on the Earth's surface is considered its variation is of the order of ω^2, where ω is the angular velocity of the Earth.

Now, expressed in radians per sidereal second

$$\omega = \frac{2\pi}{24 \cdot 60^2} = .0\,00073 + \qquad \omega^2 = .000\,000\,0053 +$$

and hence terms of the order of ω^2 may be neglected to a first approximation. The direction of ζ is, *by definition,* the vertical at A. Suppose the moving particle at any instant to have the relative coordinates ξ, η, ζ. The projections of the *turning acceleration* on the relative axes are

$$\left.\begin{aligned} J_{t_\xi} &= -2\omega\frac{d\eta}{dt}\sin\lambda \\ J_{t_\eta} &= 2\omega\left(\frac{d\xi}{dt}\sin\lambda - \frac{d\zeta}{dt}\cos\lambda\right) \\ J_{t_\zeta} &= 2\omega\frac{d\eta}{dt}\cos\lambda \end{aligned}\right\} \qquad [153]$$

where λ is the latitude of A.

161. Differential equations of motion.—Substituting these values in the differential equations of motion of Art. 159 we have

$$\left.\begin{aligned} \xi'' &= 2\omega\frac{d\eta}{dt}\sin\lambda \\ \eta'' &= -2\omega\left(\frac{d\xi}{dt}\sin\lambda - \frac{d\zeta}{dt}\cos\lambda\right) \\ \zeta'' &= g - 2\omega\frac{d\eta}{dt}\cos\lambda \end{aligned}\right\} \qquad [154]$$

The integration of these equations gives the finite equations of relative motion in terms of six arbitrary constants, which are determined by the initial position and velocity.

162. First approximation.—Neglecting the terms containing ω for a first approximation, the differential equations of motion become

$$\xi'' = \eta'' = o \qquad \zeta'' = g \qquad [155]$$

that is, the particle is acted on by an approximately constant relative force

$$F_2 = mg$$

directed along the vertical toward the interior of the Earth.

This force is termed the *weight.*

163. Second approximation.—Retaining the terms containing ω and assuming the particle to start from the origin with zero relative velocity, we have by integrating equations [154] once

$$\left.\begin{aligned} \xi' &= 2\omega\eta \sin \lambda \\ \eta' &= -2\omega(\xi \sin \lambda - \zeta \cos \lambda) \\ \zeta' &= gt - 2\omega\eta \cos \lambda \end{aligned}\right\} \qquad [156]$$

Attempting to satisfy these equations by functions of the form

$$\left.\begin{aligned} \xi &= \xi_0 + \omega\xi_1 + \omega^2\xi_2 + \\ \eta &= \eta_0 + \omega\eta_1 + \omega^2\eta_2 + \\ \zeta &= \zeta_0 + \omega\zeta_1 + \omega^2\zeta_2 + \end{aligned}\right\} \qquad [157]$$

where ξ_0, η_0, ζ_0, ξ_1, η_1, ζ, etc., are functions of t which vanish together with their first derivatives for $t = 0$, we have by substituting these developments in equations [156] and equating like powers of ω

$$\frac{d\xi_0}{dt} = 0 \qquad \frac{d\eta_0}{dt} = 0 \qquad \frac{d\zeta_0}{dt} = gt$$

whence

$$\xi_0 = \eta_0 = 0 \qquad \zeta_0 = \frac{gt^2}{2}$$

Similarly

$$\frac{d\xi_1}{dt} = 0 \qquad \frac{d\eta_1}{dt} = gt^2 \cos \lambda \qquad \frac{d\zeta_1}{dt} = 0$$

whence

$$\xi_1 = o \qquad \eta_1 = \frac{gt^3}{3} \cos \lambda \qquad \zeta_1 = o$$

Neglecting the terms in ω^2 *in the development* [157] *we have*

$$\xi = o \qquad \eta = \omega \frac{gt^3}{3} \cos \lambda \qquad \zeta = \frac{gt^2}{2} \qquad [158]$$

to the second approximation.

The particle then remains in the plane of $\eta\zeta$, but instead of falling along the vertical it is deviated slightly toward the *east.* In the $\eta\zeta$ plane the particle describes the semicubical parabola

$$3\sqrt{2g} \cdot \eta = 4\omega\zeta^{3/2} \cos \lambda$$

which gives the eastern deviation for a fall from a given height.

Retaining terms in ω^2 in [157] we should get a southern deviation, but since the variation in g is also of the order of ω^2 this method will not give its numerical value.

CHAPTER XVI

CONSTRAINED MOTION OF A PARTICLE. GENERAL CONSIDERATIONS

164. Constrained particle.—165. Mathematical expression of the conditions of constraint.—166. Variable constraints.—167. Smooth constraints.—168. Forces of constraint.—169. Motion of a constrained particle.—170. Direction of F_r.—172. Magnitude of F_r.—173. Particle at rest on a smooth surface.—174. Curve of constraint.—175. Particle at rest on a smooth curve.—176. Conditions of equilibrium.

164. **Constrained particle.**—A particle is said to be constrained if it is prevented by the physical conditions of the problem from occupying certain positions in space, or if it is prevented from passing from one position to another along certain paths. For example, a particle attached to a fixed point A by an inextensible string of length l can never reach that portion of space outside of the sphere of radius l about A as centre. Similarly a particle may be constrained to lie on a surface (S) or on a curve (C).

165. **Mathematical expression of the conditions of constraint.**—In every case the conditions of restraint may be expressed by mathematical relations among the coordinates of the particle. In the first example above the condition may be expressed by writing

$$(x-a_1)^2+(y-a_2)^2+(z-a_3)^2 \leqq l^2$$

where a_1, a_2, a_3 are the coordinates of A, and in general, if the particle is required to lie within or without a surface (S) we may write the condition that the distance of the particle from some point within (S) shall be less or greater than the line from this point to the surface.

If the particle is required to lie on the surface (S), then at every instant its coordinates must satisfy the equation of the surface

$$S=0$$

If it is required to lie on the curve (C), then its coordinates must satisfy the equations of the curve.

The *physical* conditions are thus replaced by mathematical equations.

166. Variable constraints.—If the constraining surfaces or curves change their form or position or both with the time, they are termed *variable*, and their equations will involve the time. These are excluded from the following discussion.

167. Smooth constraints.—I shall assume the constraints to leave unaltered the component of the acceleration *tangential* to them. They are then termed *smooth*, or without *friction*.

168. Forces of constraint.—In addition to the constraints the particle may be acted on by certain forces, termed the *external* forces. Let F_e be the geometric sum of these external forces and let J_e be the corresponding acceleration, so that

$$\overline{F_e}=m\cdot\overline{J_e} \qquad [159]$$

The particle under the action of F_e and the constraints will take a certain motion, and will therefore at each instant have a certain velocity and acceleration. Let V and J represent these and write

$$\overline{J}-\overline{J_e}\equiv\overline{J_R} \qquad [160]$$

I shall term J_r the *acceleration of the constraints* and F_R, where

$$\overline{F_R}=m\overline{J_R}$$

the *force of the constraints.*

169. Motion of a constrained particle.—It follows that the motion of the constrained particle will be the same as that of a free particle acted on by a force F, where

$$\overline{F}=\overline{F_e}+\overline{F_R}$$

Hence, given F_e and the constraints the problem is reduced to the determination of F_R, or conversely, given the motion and the constraints, to the determination of F and thence F_e and F_R, *in a form not involving the time explicitly.*

170. Direction of F_R. Surface of constraint.—The tangential components of J_e are unaltered. Hence

$$\overline{J_t}=\overline{J_{e_t}}$$

But $$\overline{J_t}=\overline{J_{e_t}}+\overline{J_{R_t}}$$

Therefore $$J_{R_t}=0 \qquad [161]$$

and, *the acceleration of constraint is normal to the surface of constraint.*

171. Magnitude of F_R.—Projecting on the normal

$$\overline{J_N}=\overline{J_{e_N}}+\overline{J_R}$$

Hence $$\overline{J_R}=\overline{J_N}-\overline{J_{e_N}} \qquad [162]$$

and $$\overline{F_R}=\overline{F_N}-\overline{F_{e_N}} \qquad [163]$$

172. Particle at rest on a smooth surface.—If the particle is at rest

$$J=0$$

and $$\overline{J_R}=-\overline{J_{e_N}}$$

HENCE, *the reaction of a smooth surface on a particle at rest on it is equal and opposite to the projection of the resultant of the external forces on the normal.*

173. Curve of constraint.—As before, the tangential component of J_e is unaltered. Hence

$$J_t=J_{e_t}+J_{R_t}=J_{e_t}$$

Whence $$J_{R_t}=0$$

and therefore, *the reaction is in the normal plane to the curve.*

Projecting on the normal and the binormal

$$\overline{J_N}=\overline{J_{e_N}}+\overline{J_{R_N}}$$

$$\overline{J_B}=\overline{J_{e_B}}+\overline{J_{R_B}}=0$$

whence

$$\left.\begin{aligned}\overline{J_{R_N}}&=\overline{J_N}-\overline{J_{e_N}}\\ \overline{J_{R_B}}&=-\overline{J_{e_B}}\end{aligned}\right\}\qquad[164]$$

which give the components along the normal and binormal and hence determine J_R.

174. Particle at rest on a smooth curve.—If the particle is at rest, then

and

$$\left.\begin{aligned}J&=0=J_{e_t}\\ \overline{J_{R_N}}&=-\overline{J_{e_N}}\\ J_{R_B}&=-J_{e_B}\end{aligned}\right\}\qquad[165]$$

THEREFORE, *for a particle at rest on a smooth curve the reaction is equal and opposite to the external force.*

175. Conditions of equilibrium.—Hence, in order that a particle be in equilibrium on a smooth surface the external force must be diverted along the normal, and in order that it be in equilibrium on a smooth curve the external force must lie in the normal plane.

CHAPTER XVII

CONSTRAINED MOTION ON EARTH'S SURFACE

176. Variation of g with latitude.—177. Particle falling on smooth curve.—178. Oscillations of a simple pendulum.—179. Oscillations of small amplitude.—180. Velocity in oscillations of any amplitude.—181. Tension in the string.—182. Foucault's pendulum.—183. Approximate solution. Particular case. General case. Foucault's case. Theorem of Chevilliet.

176. Variation of g with latitude.—In Art. 160 the direction of the vertical at a point A was defined as the direction of the geometrical difference of the absolute and convective acceleration of A, and the magnitude of this difference was denoted by g, termed the *acceleration of gravity at A*. This acceleration is then defined by the equipollence

$$\bar{g} = \overline{J_a} - \overline{J_c} \qquad [166]$$

Neglecting, for a first approximation, the attraction of the sun and the moon and the orbital motion of the Earth about the Sun, and assuming the Earth to be a sphere composed of concentric homogeneous layers, J_a is at any instant directed along the line joining A with the centre O of the Earth. Denote this value of J_a by g_0, and choose the absolute axes as follows: X and Y in the plane of the equator, and Z through O, coincident with the axis of rotation of the Earth, positive direction toward the north. Orient the relative axes as in Art. 160, that is, ζ along the direction of g, Ξ in the meridian toward the south, and η toward the east. Denote by ω the angular velocity of the Earth and by a its radius, and let r (fig. 36) be the radius of the parallel on which A lies. Then J_c is constantly directed along r and its magnitude is

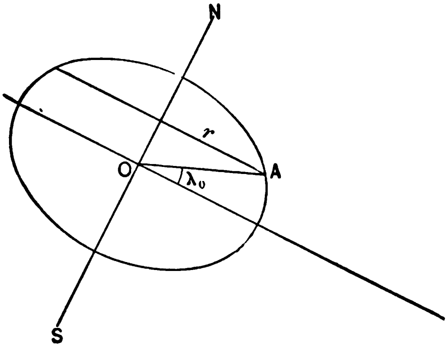


FIG. 36.

$$J_c = r\omega^2 = a\omega^2 \cos \lambda_0 \qquad [167]$$

where λ_0 is the geocentric latitude of A. Denoting, as before, by λ the astronomical latitude of A, and projecting the equipollence

$$\overline{g} = \overline{g_0} - \overline{a\omega^2 \cos \lambda_0}$$

on $A\zeta$, we have

$$g = g_0 \cos (\lambda - \lambda_0) - a\omega^2 \cos \lambda_0 \cos \lambda \qquad [168]$$

The angle $\lambda - \lambda_0$, termed the *equation of the vertical,* may to a first approximation be taken as zero, whence $$g = g_0 - a\omega^2 \cos^2 \lambda \qquad [169]$$

From observations

$$g = 980.6059 \text{ cm.} - 2.5028 \text{ cm.} \cos^2 \lambda \qquad [170]$$

177. Particle falling on a smooth curve.—If a particle M is constrained to describe a curve (C) under the action of an external force F_e, obeying a known law, the problem is to determine the equation

$$s = f(t)$$

giving the arc, measured from some initial point, in terms of the time.

The equation of motion may also be determined in the form

$$t = \phi(s)$$

where it is solved for t.

As we have seen, the reaction N of the constraint is always in the normal plane to the curve, and M may be regarded as free under the action of the two forces N and F_e.

Since the tangential acceleration is unaltered

$$mJ_t = F_{e_t}$$

or $$m\frac{d^2s}{dt^2} = F_{e_t} \qquad [171]$$

which determines s in terms of t when the initial conditions are known.

If the external force is the force of gravity and the relative axes be orientated with Z drawn along the vertical *from* the interior of the Earth (fig. 37), and if V be the velocity of M at any point, and θ the angle between the tangent at M and the axis AZ, then

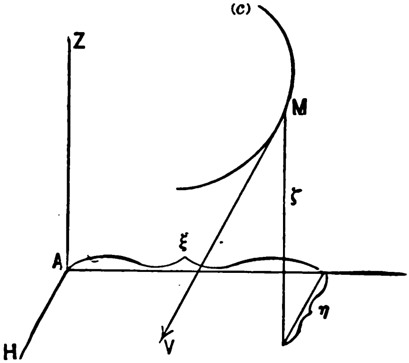


FIG. 37.

$$\cos\theta=\frac{\zeta'}{V}$$

whence

$$Fe_t=-mg\frac{\zeta'}{V}$$

or

$$mV'=-mg\frac{\zeta'}{V}$$

Hence

$$VV'=-g\zeta'$$

or

$$VV'+g\zeta'=0$$

which may be written

$$\left(\frac{V^2}{2}+g\zeta\right)'=0$$

Integrating

$$\frac{V^2}{2}+g\zeta=c \quad \text{a constant}$$

If then ζ_0 is the initial altitude and V_0 the initial velocity,

$$\frac{V^2}{2}+g\zeta=\frac{V_0^2}{2}+g\zeta_0$$

or

$$V^2=V_0^2+2g(\zeta_0-\zeta) \qquad [172]$$

which gives the velocity of M when it passes any point of its path of altitude ζ.

It is to be noted that this velocity depends only on V_0 and $\zeta_0-\zeta$, that is, on the initial velocity and the difference in altitude of the given point and the initial point, and not at all on the form of the curve (C).

Since (C) is supposed given, then ζ is a known function of s, and hence

$$V=\pm\sqrt{V_0^2+2g(\zeta_0-\zeta)}=\phi_1(s)$$

whence

$$\frac{ds}{dt}=\phi_1(s)$$

or

$$dt=\frac{ds}{\phi_1(s)}$$

Integrating
$$t=\int_{s_0}^{s}\frac{ds}{\phi_1(s)}=\phi(s)+C$$

$$=\phi(s)-\phi(s_0) \qquad [173]$$

and the problem is solved.

178. **Oscillations of a simple pendulum.**—A simple pendulum is a particle M constrained to describe a circumference of radius $l=AM$ under the action of gravity (fig. 38).

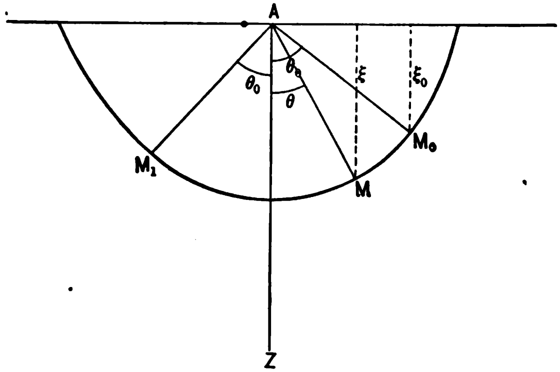


FIG. 38.

Suppose the relative axes orientated with AZ along the vertical *toward* the interior of the Earth, and $A\Xi$ in the plane of AZ and M_0, the initial position of the pendulum. *If we neglect the rotation of the Earth,* there is the constant force

$$F_e=mg$$

acting parallel to AZ, and hence the pendulum remains in the plane ΞAZ.

Let θ be the angle which AM makes with AZ at any instant t, and let the pendulum be at its highest position at the instant $t=0$. Count s from the lowest position, positive when ξ is positive.

Then $$s=l\theta \qquad V=l\theta' \qquad V'=l\theta''$$

$$\zeta=l\cos\theta \qquad \zeta'=-l\sin\theta\cdot\theta'$$

whence $$l^2\cdot\theta'\cdot\theta''=-gl\sin\theta\cdot\theta'$$

or $$\theta''=-\frac{g}{l}\sin\theta \qquad [174]$$

This is the differential equation of the motion.

Integrating once

$$\frac{1}{2}\theta'^2=\frac{g}{l}\cos\theta+c$$

Now when $\theta=\theta_0$, then $\theta'=\theta_0'$, and hence

$$c=\frac{1}{2}\theta_0'^2-\frac{g}{l}\cos\theta_0$$

Therefore $$\theta'=\sqrt{\theta_0'^2+\frac{2g}{l}(\cos\theta-\cos\theta_0)} \qquad [175]$$

whence $$dt=\frac{d\theta}{\sqrt{\theta_0'^2+\frac{2g}{l}(\cos\theta-\cos\theta_0)}}$$

or $$t=\int\frac{d\theta}{\sqrt{\theta_0'^2+\frac{2g}{l}(\cos\theta-\cos\theta_0)}}+c \qquad [176]$$

179. Oscillations of small amplitude.—The integration of this expression leads to *elliptic* functions, but a simplification may be introduced if we assume θ_0 so small that, for a first approximation, $\sin\theta$ may be replaced by θ in [174]. In this case the differential equation of motion becomes

$$\theta'' = -\frac{g}{l}\theta$$

and the motion is harmonic.

Integrating

$$\theta = A\cos\sqrt{\frac{g}{l}}\cdot t + B\sin\sqrt{\frac{g}{l}}\cdot t \qquad [177]$$

whence $$V = l\theta' = l\sqrt{\frac{g}{l}}\left[-A\sin\sqrt{\frac{g}{l}}\cdot t + B\cos\sqrt{\frac{g}{l}}\cdot t\right]$$

If the pendulum is abandoned at rest, then for $t=0$ we must have $V=0$, whence $B=0$, and

$$\theta = A\cos\sqrt{\frac{g}{l}}\cdot t$$

Since for $t=0$, $\theta=\theta_0$, we have

$$\left.\begin{aligned}\theta &= \theta_0\cos\sqrt{\frac{g}{l}}\cdot t\\ \text{and}\quad \theta' &= -\theta_0\sqrt{\frac{g}{l}}\cdot\sin\sqrt{\frac{g}{l}}\cdot t\end{aligned}\right\} \qquad [178]$$

The period T is such that replacing t by $t+T$ both θ and θ' take their original values,

whence
$$\sqrt{\frac{g}{l}}\cdot T=2\pi$$

or
$$T=2\pi\sqrt{\frac{l}{g}}$$

180. Velocity in oscillations of any amplitude.—Substituting $\zeta=l\cos\theta$ in [172] we have

$$V^2=V_0{}^2+2gl(\cos\theta-\cos\theta_0)\,. \qquad [179]$$

which agrees with [175], and gives the velocity at any point of the path for any amplitude.

181. Tension in the string.—From the equations of Art. 168

$$J_{R_N}=J_N-J_{e_N}$$

$$J_{R_B}=-J_{e_B}$$

we have at once

$$J_{R_N}=\frac{V^2}{l}+g\cos\theta$$

$$J_{R_B}=0$$

Hence the tension in the string is

$$T = m\frac{V^2}{l} + mg\cos\theta$$

$$= m\frac{V_0^2}{l} + mg(3\cos\theta - \cos\theta_0) \qquad [180]$$

The maximum value of T is when the pendulum passes through the lowest point of its path, or

$$T_{\text{maximum}} = \frac{mV_0^2}{l} + mg(3 - \cos\theta_0) \qquad [181]$$

182. Foucault's pendulum.—If the rotation of the Earth be taken into account, then the external relative force is

$$\left.\begin{aligned} F_{re_\xi} &= -mJ_{t_\xi} \\ F_{re_\eta} &= -mJ_{t_\eta} \\ F_{re_\zeta} &= mg - mJ_{t_\zeta} \end{aligned}\right\}$$

The particle is constrained to lie on the sphere

$$\xi^2 + \eta^2 + \zeta^2 = l^2$$

and calling N the reaction of this *smooth* surface, we have

$$N_\xi = -N\cdot\frac{\xi}{l} \qquad N_\eta = -N\cdot\frac{\eta}{l} \qquad N_\zeta = -N\cdot\frac{\zeta}{l}$$

since N is directed toward A.

The relative force acting is then

$$\left.\begin{aligned} F_{r_\xi} &= -mJ_{t_\xi} - N\cdot\frac{\xi}{l} \\ F_{r_\eta} &= -mJ_{t_\eta} - N\cdot\frac{\eta}{l} \\ F_{r_\zeta} &= mg - mJ_{t_\zeta} - N\cdot\frac{\zeta}{l} \end{aligned}\right\}$$

and the differential equations of relative motion are

$$\left.\begin{aligned} m\xi'' &= -mJ_{t_\xi} - N\cdot\frac{\xi}{l} \\ m\eta'' &= -mJ_{t_\eta} - N\cdot\frac{\eta}{l} \\ m\zeta'' &= mg - mJ_{t_\zeta} - N\cdot\frac{\zeta}{l} \end{aligned}\right\}$$

Replacing the components of the turning acceleration by the expressions of Art. 161 we have finally

$$\left.\begin{aligned} m\xi'' &= -2m\omega\left(-\frac{d\eta}{dt}\sin\lambda + \frac{d\zeta}{dt}\cos\lambda\right) - N\cdot\frac{\xi}{l} \\ m\eta'' &= -2m\omega\frac{d\xi}{dt}\sin\lambda - N\cdot\frac{\eta}{l} \\ m\zeta'' &= mg + 2m\omega\frac{d\xi}{dt}\cos\lambda - N\cdot\frac{\zeta}{l} \end{aligned}\right\} \quad [182]$$

These three simultaneous differential equations together with the equation of constraint

$$\xi^2 + \eta^2 + \zeta^2 = l^2 \quad [183]$$

give the motion.

In order to integrate [182] the reaction of constraint N must be eliminated by means of [183]. This may be done as follows:

Solving [183] for ζ we have

$$\zeta = \sqrt{l^2 - \xi^2 - \eta^2}$$

Differentiating

$$\zeta' = (\sqrt{l^2 - \xi^2 - \eta^2})' \qquad [184]$$

$$\zeta'' = (\sqrt{l^2 - \xi^2 - \eta^2})''$$

Substituting in the third equation of [110] we have

$$m(\sqrt{l^2 - \xi^2 - \eta^2})'' = mg - 2m\omega\eta' \cos \lambda - N\frac{\sqrt{l^2 - \xi^2 - \eta^2}}{l}$$

Whence

$$N = \frac{ml}{\sqrt{l^2 - \xi^2 - \eta^2}}\{g - 2\omega\eta' \cos \lambda - (\sqrt{l^2 - \xi^2 - \eta^2})''\}$$

Substituting these values of ζ and N in the first two equations of [182] we have

$$\left.\begin{aligned} \xi'' &= 2\omega\eta' \sin \lambda - \frac{\xi}{\sqrt{l^2 - \xi^2 - \eta^2}}\{g - 2\omega\eta' \cos \lambda \\ &\qquad - (\sqrt{l^2 - \xi^2 - \eta^2})''\} \\ \eta'' &= -2\omega\{\xi' \sin \lambda - (\sqrt{l^2 - \xi^2 - \eta^2})' \cos \lambda\} \\ &\quad - \frac{\eta}{\sqrt{l^2 - \xi^2 - \eta^2}}\{g - 2\omega\eta \cos \lambda - (\sqrt{l^2 - \xi^2 - \eta^2})''\} \end{aligned}\right\} [185]$$

two simultaneous differential equations in ξ and η only. The integration of these gives ξ and η as functions of t; then by substitution in [184] ξ as a function of t is obtained and the problem is *theoretically* solved. The mathematical difficulties, however, are such as to prevent this method of procedure, and leaving aside the general case we may obtain an approximate solution as follows:

183. Approximate solution.—Regard $\frac{\xi}{l}$ and $\frac{\eta}{l}$ and their derivatives, together with ω, as *small quantities* whose *squares* and *products* may be neglected. Then from

$$\zeta = l\left(1 - \frac{\xi^2 + \eta^2}{l^2}\right)^{\frac{1}{2}}$$

we have $\zeta = l$, and equations [185] become

$$\left.\begin{aligned} \xi'' &= 2\omega \sin \lambda \cdot \eta' - \frac{g}{l} \cdot \xi \\ \eta'' &= -2\omega \sin \lambda \cdot \xi' - \frac{g}{l} \cdot \eta \end{aligned}\right\} \qquad [186]$$

These are linear equations with constant coefficients and can be integrated by quadratures. The following method will lead to interesting results.

Project the motion on the plane

$$\zeta = l$$

which is tangent to the sphere of constraint at its lowest point A_1. Projecting $A\xi$ and $A\eta$ on this plane, giving the axes $A_1\xi$ and $A_1\eta$ (fig. 39), then the differential equations of motion of P_1, the projected pendulum, are

$$\left.\begin{aligned}\xi'' &= 2\omega\sin\lambda\eta' - \frac{g}{l}\xi\\ \eta'' &= -2\omega\sin\lambda\xi' - \frac{g}{l}\eta\end{aligned}\right\} . \qquad [187]$$

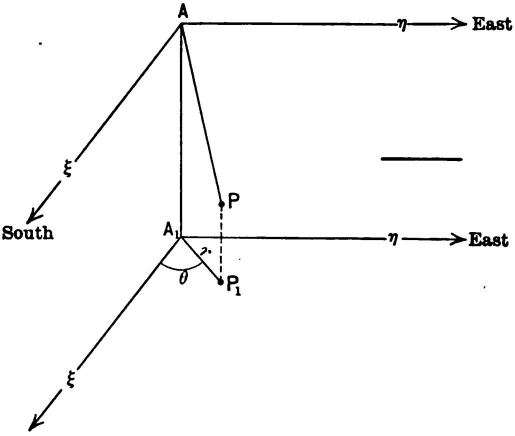


FIG. 39.

From which

$$\left.\begin{aligned}(\xi'\xi''+\eta'\eta'') + \frac{g}{l}(\xi\xi'+\eta\eta') &= 0\\ (\xi\eta''-\eta\xi'') + 2\omega\sin\lambda(\xi\xi'+\eta\eta') &= 0\end{aligned}\right\} . \qquad [188]$$

These are exact equations, and integrating we have

$$\left.\begin{aligned}(\xi'^2+\eta'^2) + \frac{g}{l}(\xi^2+\eta^2) &= h\\ (\xi\eta'-\eta\xi') + \omega\sin\lambda(\xi^2+\eta^2) &= C\end{aligned}\right\} \qquad [189]$$

Calling r and θ the polar coordinates of the projected pendulum

$$\xi = r\cos\theta \qquad \eta = r\sin\theta$$

and [189] become

$$\left.\begin{aligned} r'^2 + r^2\theta'^2 + \frac{g}{l}r^2 = h \\ r^2\theta' + \omega\sin\lambda\cdot r^2 = C \end{aligned}\right\} \qquad [190]$$

Particular case.—Suppose the pendulum in equilibrium in the vertical and set it in motion by a small impulse.

Since r and θ are both zero at the initial instant, then C is zero and the second equation of [190] becomes

$$\theta' + \omega\sin\lambda = 0$$

whence

$$\theta = \theta_0 - \omega\sin\lambda\cdot t$$

and the projected pendulum revolves about the origin, with constant angular velocity

$$\omega_1 = \omega\sin\lambda$$

in the negative direction. It makes a complete revolution in the time

$$T = \frac{2\pi}{\omega_1} = \frac{2\pi}{\omega\sin\lambda} = \frac{24^{\mathrm{h}}}{\sin\lambda} \quad \text{(sidereal time)}$$

Hence

$$T_{\text{St. Louis}} = 38^{\mathrm{h}}.44 \text{ (about)}$$

The pendulum then seems to oscillate in a plane which rotates about the vertical in the negative direction with con-

stant angular velocity $\omega_1=\omega \sin \lambda$. *The time of a complete revolution is* $\dfrac{24}{\sin \lambda}$ *sidereal hours.*

General case.—Introducing the new variable

$$\theta_1=\theta+\omega \sin \lambda \cdot t=\theta+\omega_1 t$$

equations [190] become

$$\left.\begin{aligned} r'^2+r^2\left(\theta_1'^2+\omega_1^2-2\omega_1\theta_1'+\frac{g}{l}\right)&=h \\ r^2\theta_1'&=C \end{aligned}\right\} \qquad [191]$$

The variables r *and* θ_1 *are the polar coordinates of* P_1 *referred to a set of axes* $A_1\xi_1$ *and* $A_1\eta_1$ *which rotate about the vertical in the negative direction with constant angular velocity* ω_1.

Neglecting the term containing ω_1^2 and calling h_1 a new constant equations [191] become

$$\left.\begin{aligned} r'^2+r^2\theta_1'^2+\frac{g}{l}\cdot r^2&=h_1 \\ r^2\theta_1'&=C \end{aligned}\right\} \qquad [192]$$

These may be written

$$\left.\begin{aligned} V^2+\frac{g}{l}\cdot r^2&=h_1 \\ \text{Areal } V&=2C \end{aligned}\right\} \qquad [193]$$

Hence referred to these axes the projected pendulum moves under the action of a force $\frac{g}{l}\cdot r$ directed toward A_1. The path is then an ellipse of period

$$T_1 = 2\pi\sqrt{\frac{l}{g}} \qquad [195]$$

The projected pendulum then describes an ellipse in the horizontal plane, and the major axis of this ellipse rotates about the vertical in the negative direction with angular velocity ω_1.

Foucault's case.—In Foucault's celebrated experiment in the Pantheon at Paris, the pendulum was drawn aside from the vertical and the restraining string burned. The initial velocity relative to the axes $A_1\xi$ and $A_1\eta$ is then zero, and hence r' and θ' are both initially zero. Since $r' = 0$ initially this value of r is a maximum or a minimum and is therefore a semi-axis of the ellipse. Call it a. Then from the second equation of [190]

$$\omega \sin \lambda \cdot a^2 = \omega_1 \cdot a^2 = C$$

Then from the second equation of [192],

$$r^2\theta_1' = \omega_1 a^2$$

or

$$\text{Areal } V = 2\omega_1 \cdot a^2 \qquad [196]$$

Theorem of Chevilliet.—Let a and b be the two axes of the ellipse. The areal velocity is then

$$\text{Areal } V = \frac{\pi ab}{T_1}$$

Hence $$2\omega_1 a^2 = \frac{\pi ab}{T_1}$$

Whence $$\frac{b}{a} = \frac{T_1\omega_1}{2\pi} = \frac{T_1}{T} = \omega \sin \lambda \sqrt{\frac{l}{g}}$$

In the experiment in the Pantheon

$$l=67^{\mathrm{m}}, \quad a=3^{\mathrm{m}}, \quad T_1=16^{\mathrm{s}}, \quad T=32^{\mathrm{h}}, \quad \frac{b}{a}=\frac{1}{7200}$$

and hence the ellipse was so flat as to seem a line.

At St. Louis, with the same values of l and a we should have

$$\frac{b}{a}=\frac{16^{\mathrm{s}}.43}{38^{\mathrm{h}}.44}=\frac{1}{8423}$$

New York Public Library,
WEBSTER BRANCH,
538 East 76th Street,
CIRCULATING DEPARTMENT
CENTRAL COLLECTION

INDEX.

ARTICLE

Absolute acceleration 45, 100
Absolute axes 96, 106
Absolute coordinates 96
Absolute displacement 45, 96, 106
Absolute motion 96
Acceleration in curvilinear motion 61
Acceleration in rectilinear motion 30
Amplitude 41
Analytic expression of components of a vector 8
Analytic expression of force between two free particles 140
Analytic expression of geometric difference 6
Analytic expression of geometric sum 4
Analytic expression of geometric sum referred to axes 10
Apparent motion 48
Areal velocity 72
Argument 41
Attraction toward a fixed centre 154

Centimetre-Gram-Second system of units 133
Central forces 141
Change of units 23
Composition of forces 122
Composition of harmonic angular motions 81
Composition of harmonic rectilinear motions 51
Composition of motions along same path 66
Constant field of force 111
Constrained particle 164
Convective acceleration, 49, 100

ARTICLE
Convective motion. 49, 100
Convective velocity. 49, 100
Curve of constraint. 173

Decomposition of a vector. 7
Decomposition of forces. 123
Derived units. 131
Differential equations of motion. 144
Displacement. 55

Elongation. 41
English system of units. 135
Equations of absolute and relative motion. 97
Equilibrium of a constrained particle. 172
Equilibrium of a free particle. 124
Equipollence. 2

Field of force. 109
Finite and differential equations of motion. 144
First principle of mechanics. 107
Fixed and moving axes. 96
Force. 120
Forces of constraint. 168
Force producing extinction. 151
Force producing harmonic motion. 149
Foucault's pendulum. 182
Fundamental units. 131

Geometric derivative. 11
Geometric difference. 5
Geometric sum. 3
Gravitation. 142

Harmonic angular motion. 79
Harmonic motion. 40
Hodograph. 61
Homogeneity. 74
Homogeneous equations. 22

Isolated particle. 100

Knots. 28

ARTICLE
Mass. 119
Material point. 101
Metre-Kilogram-Second system of units. 134
Moving axes. 96

Nautical miles. 28

Oscillations of small amplitude. 179

Particle. 101
Principles of mechanics. 105
Principle of the equality of action and reaction. 139
Problem of mechanics. 103
Problems of relative motion. 157
Projection of acceleration on binormal. 64
Projection of acceleration on tangent and normal. 63
Projection of geometric derivative. 9
Projection of geometric derivative on vector. 13
Projection of velocity on displacement. 58

Resistance of particle on field. 115
Relation between linear and angular velocity. 69
Relation between linear and angular acceleration. 71
Relative acceleration. 47, 96
Relative angular motion. 80
Relative axes. 96
Relative coordinates. 96
Relative displacement. 45, 96
Relative force. 156
Relative motion. 45, 66, 96
Relative velocity. 47, 100
Resistance producing extinction. 151

Second principle of mechanics. 113
Simple pendulum. 178
Simultaneous motions. 50
Smooth constraints. 167
Superposition of fields of force. 112
Systems of units. 132

Tangential and normal force. 127
Tension in string of simple pendulum. 187

ARTICLE
Theorem of Chevilliet. 183
Third principle of mechanics. 116
Turning acceleration. 100

Uniform field of force. 110
Unit of force. 137
Units of astronomy. 136
Units of kinematics. 129
Units of mechanics. 130
Universal gravitation. 142

Variable constraints. 166
Variation of g with latitude. 176
Vectors. 1
Velocity. 56
Velocity in oscillations of any amplitude. 180

York Public Library
CENTRAL COLLECTION
WEBSTER BRANCH
538 East 76th Street
CIRCULATING DEPARTMENT

176a

SHORT-TITLE CATALOGUE

OF THE

PUBLICATIONS

OF

JOHN WILEY & SONS,

NEW YORK.

LONDON: CHAPMAN & HALL, LIMITED.

ARRANGED UNDER SUBJECTS.

Descriptive circulars sent on application. Books marked with an asterisk are sold at *net* prices only, a double asterisk (**) books sold under the rules of the American Publishers' Association at *net* prices subject to an extra charge for postage. All books are bound in cloth unless otherwise stated.

AGRICULTURE.

Armsby's Manual of Cattle-feeding	12mo,	$1 75
Principles of Animal Nutrition	8vo,	4 00
Budd and Hansen's American Horticultural Manual:		
Part I.—Propagation, Culture, and Improvement	12mo,	1 50
Part II.—Systematic Pomology	12mo,	1 50
Downing's Fruits and Fruit-trees of America	8vo,	5 00
Elliott's Engineering for Land Drainage	12mo,	1 50
Practical Farm Drainage	12mo,	1 00
Green's Principles of American Forestry	12mo,	1 50
Grotenfelt's Principles of Modern Dairy Practice. (Woll.)	12mo,	2 00
Kemp's Landscape Gardening	12mo,	2 50
Maynard's Landscape Gardening as Applied to Home Decoration	12mo,	1 50
Sanderson's Insects Injurious to Staple Crops	12mo,	1 50
Insects Injurious to Garden Crops. (*In preparation.*)		
Insects Injuring Fruits. (*In preparation.*)		
Stockbridge's Rocks and Soils	8vo,	2 50
Woll's Handbook for Farmers and Dairymen	16mo.	1 50

ARCHITECTURE.

Baldwin's Steam Heating for Buildings	12mo,	2 50
Berg's Buildings and Structures of American Railroads	4to,	5 00
Birkmire's Planning and Construction of American Theatres	8vo,	3 00
Architectural Iron and Steel	8vo,	3 50
Compound Riveted Girders as Applied in Buildings	8vo,	2 00
Planning and Construction of High Office Buildings	8vo,	3 50
Skeleton Construction in Buildings	8vo,	3 00
Briggs's Modern American School Buildings	8vo,	4 00
Carpenter's Heating and Ventilating of Buildings	8vo,	4 00
Freitag's Architectural Engineering. 2d Edition, Rewritten	8vo,	3 50
Fireproofing of Steel Buildings	8vo,	2 50
French and Ives's Stereotomy	8vo,	2 50
Gerhard's Guide to Sanitary House-inspection	16mo,	1 00
Theatre Fires and Panics	12mo,	1 50
Holly's Carpenters' and Joiners' Handbook	18mo,	75
Johnson's Statics by Algebraic and Graphic Methods	8vo,	2 00

Kidder's Architect's and Builder's Pocket-book. Rewritten Edition.	**16mo, mor.,**	5 00
Merrill's Stones for Building and Decoration	8vo,	5 00
Non-metallic Minerals: Their Occurrence and Uses	8vo,	4 00
Monckton's Stair-building	4to,	4 00
Patton's Practical Treatise on Foundations	8vo,	5 00
Peabody's Naval Architecture.	8vo,	7 50
Richey's Handbook for Superintendents of Construction. (*In press.*)		
Sabin's Industrial and Artistic Technology of Paints and Varnish	8vo,	3 00
Siebert and Biggin's Modern Stone-cutting and Masonry	8vo,	1 50
Snow's Principal Species of Wood	8vo,	3 50
Sondericker's Graphic Statics with Applications to Trusses, Beams, and Arches.	8vo,	2 00
Towne's Locks and Builders' Hardware.	18mo, morocco,	3 00
Wait's Engineering and Architectural Jurisprudence	8vo,	6 00
	Sheep,	6 50
Law of Operations Preliminary to Construction in Engineering and Architecture	8vo,	5 00
	Sheep,	5 50
Law of Contracts	8vo,	3 00
Wood's Rustless Coatings: Corrosion and Electrolysis of Iron and Steel	**8vo,**	4 00
Woodbury's Fire Protection of Mills	8vo,	2 50
Worcester and Atkinson's Small Hospitals, Establishment and Maintenance, Suggestions for Hospital Architecture, with Plans for a Small Hospital.	12mo,	1 25
The World's Columbian Exposition of 1893	**Large 4to,**	1 00

ARMY AND NAVY.

Bernadou's Smokeless Powder, Nitro-cellulose, and the Theory of the Cellulose Molecule	**12mo,**	2 50
* Bruff's Text-book Ordnance and Gunnery	8vo,	6 00
Chase's Screw Propellers and Marine Propulsion	8vo,	3 00
Craig's Azimuth	4to,	3 50
Crehore and Squire's Polarizing Photo-chronograph	8vo,	3 00
Cronkhite's Gunnery for Non-commissioned Officers	**24mo, morocco,**	2 00
* Davis's Elements of Law	8vo,	2 50
* Treatise on the Military Law of United States	8vo,	7 00
	Sheep,	7 50
De Brack's Cavalry Outpost Duties. (Carr.)	**24mo morocco,**	2 00
Dietz's Soldier's First Aid Handbook	**16mo, morocco,**	1 25
* **Dredge's Modern French Artillery**	**4to, half morocco,**	15 00
Durand's Resistance and Propulsion of Ships	**8vo,**	5 00
* **Dyer's Handbook of Light Artillery**	**12mo,**	3 00
Eissler's Modern High Explosives	8vo,	4 00
* Fiebeger's Text-book on Field Fortification	Small 8vo,	2 00
Hamilton's The Gunner's Catechism	**18mo,**	1 00
* Hoff's Elementary Naval Tactics	**8vo,**	1 50
Ingalls's Handbook of Problems in Direct Fire	**8vo,**	4 00
* Ballistic Tables	**8vo,**	1 50
* **Lyons's Treatise on Electromagnetic Phenomena. Vols. I. and II.**	**8vo. each,**	6 00
* **Mahan's Permanent Fortifications. (Mercur.)**	**8vo, half morocco,**	7 50
Manual for Courts-martial	16mo, morocco,	1 50
* **Mercur's Attack of Fortified Places**	**12mo,**	2 00
* Elements of the Art of War	**8vo,**	4 00
Metcalf's Cost of Manufactures—And the Administration of Workshops, Public and Private	**8vo,**	5 00
* Ordnance and Gunnery. 2 vols.	**12mo,**	5 00
Murray's Infantry Drill Regulations	**18mo, paper,**	10
Nixon's Adjutants' Manual	24mo,	1 00
Peabody's Naval Architecture.	**8vo,**	7 50

* Phelps's Practical Marine Surveying	8vo,	2 50
Powell's Army Officer's Examiner	12mo,	4 00
Sharpe's Art of Subsisting Armies in War	18mo, morocco,	1 50
* Walke's Lectures on Explosives	8vo,	4 00
* Wheeler's Siege Operations and Military Mining	8vo,	2 00
Winthrop's Abridgment of Military Law	12mo,	2 50
Woodhull's Notes on Military Hygiene	16mo,	1 50
Young's Simple Elements of Navigation	16mo morocco,	1 00
Second Edition, Enlarged and Revised	16mo, morocco,	2 00

ASSAYING.

Fletcher's Practical Instructions in Quantitative Assaying with the Blowpipe.	12mo, morocco,	1 50
Furman's Manual of Practical Assaying	8vo,	3 00
Lodge's Notes on Assaying and Metallurgical Laboratory Experiments	8vo,	3 00
Miller's Manual of Assaying	12mo,	1 00
O'Driscoll's Notes on the Treatment of Gold Ores	8vo,	2 00
Ricketts and Miller's Notes on Assaying	8vo,	3 00
Ulke's Modern Electrolytic Copper Refining	8vo,	3 00
Wilson's Cyanide Processes	12mo,	1 50
Chlorination Process	12mo,	1 50

ASTRONOMY.

Comstock's Field Astronomy for Engineers	8vo,	2 50
Craig's Azimuth	4to,	3 50
Doolittle's Treatise on Practical Astronomy	8vo,	4 00
Gore's Elements of Geodesy	8vo,	2 50
Hayford's Text-book of Geodetic Astronomy	8vo,	3 00
Merriman's Elements of Precise Surveying and Geodesy	8vo,	2 50
* Michie and Harlow's Practical Astronomy	8vo,	3 00
* White's Elements of Theoretical and Descriptive Astronomy	12mo,	2 00

BOTANY.

Davenport's Statistical Methods, with Special Reference to Biological Variation.	16mo, morocco,	1 25
Thomé and Bennett's Structural and Physiological Botany	16mo,	2 25
Westermaier's Compendium of General Botany. (Schneider.)	8vo,	2 00

CHEMISTRY.

Adriance's Laboratory Calculations and Specific Gravity Tables	12mo,	1 25
Allen's Tables for Iron Analysis	8vo,	3 00
Arnold's Compendium of Chemistry. (Mandel.)	Small 8vo,	3 50
Austen's Notes for Chemical Students	12mo,	1 50
* Austen and Langworthy. The Occurrence of Aluminium in Vegetable Products, Animal Products, and Natural Waters	8vo,	2 00
Bernadou's Smokeless Powder.—Nitro-cellulose, and Theory of the Cellulose Molecule	12mo,	2 50
Bolton's Quantitative Analysis	8vo,	1 50
* Browning's Introduction to the Rarer Elements	8vo,	1 50
Brush and Penfield's Manual of Determinative Mineralogy	8vo,	4 00
Classen's Quantitative Chemical Analysis by Electrolysis. (Boltwood.)	8vo,	3 00
Cohn's Indicators and Test-papers	12mo,	2 00
Tests and Reagents	8vo,	3 00
Craft's Short Course in Qualitative Chemical Analysis. (Schaeffer.)	12mo,	1 50
Dolezalek's Theory of the Lead Accumulator (Storage Battery). (Von Ende)	12mo,	2 50
Drechsel's Chemical Reactions. (Merrill.)	12mo,	1 25
Duhem's Thermodynamics and Chemistry. (Burgess.)	8vo,	4 00
Eissler's Modern High Explosives	8vo,	4 00
Effront's Enzymes and their Applications. (Prescott.)	8vo,	3 00

Erdmann's Introduction to Chemical Preparations. (Dunlap.) 12mo,	1 25
Fletcher's Practical Instructions in Quantitative Assaying with the Blowpipe 12mo, morocco,	1 50
Fowler's Sewage Works Analyses 12mo,	2 00
Fresenius's Manual of Qualitative Chemical Analysis. (Wells.) 8vo,	5 00
Manual of Qualitative Chemical Analysis. Part I. Descriptive. (Wells.) 8vo,	3 00
System of Instruction in Quantitative Chemical Analysis. (Cohn.) 2 vols. 8vo,	12 50
Fuertes's Water and Public Health 12mo,	1 50
Furman's Manual of Practical Assaying 8vo,	3 00
Getman's Exercises in Physical Chemistry 12mo,	
Gill's Gas and Fuel Analysis for Engineers 12mo,	1 25
Grotenfelt's Principles of Modern Dairy Practice. (Woll.) 12mo,	2 00
Hammarsten's Text-book of Physiological Chemistry. (Mandel.) 8vo,	4 00
Helm's Principles of Mathematical Chemistry. (Morgan.) 12mo,	1 50
Hering's Ready Reference Tables (Conversion Factors) 16mo, morocco,	2 50
Hinds's Inorganic Chemistry 8vo,	3 00
* Laboratory Manual for Students 12mo,	75
Holleman's Text-book of Inorganic Chemistry. (Cooper.) 8vo,	2 50
Text-book of Organic Chemistry. (Walker and Mott.) 8vo,	2 50
* Laboratory Manual of Organic Chemistry. (Walker.) 12mo,	1 00
Hopkins's Oil-chemists' Handbook 8vo,	3 00
Jackson's Directions for Laboratory Work in Physiological Chemistry 8vo,	1 25
Keep's Cast Iron 8vo,	2 50
Ladd's Manual of Quantitative Chemical Analysis 12mo,	1 00
Landauer's Spectrum Analysis. (Tingle.) 8vo,	3 00
Lassar-Cohn's Practical Urinary Analysis. (Lorenz.) 12mo,	1 00
Application of Some General Reactions to Investigations in Organic Chemistry. (Tingle.) 12mo,	1 00
Leach's The Inspection and Analysis of Food with Special Reference to State Control. 8vo,	7 50
Löb's Electrolysis and Electrosynthesis of Organic Compounds. (Lorenz.) 12mo,	1 00
Lodge's Notes on Assaying and Metallurgical Laboratory Experiments 8vo,	3 00
Lunge's Techno-chemical Analysis. (Cohn.) 12mo,	1 00
Mandel's Handbook for Bio-chemical Laboratory 12mo,	1 50
* Martin's Laboratory Guide to Qualitative Analysis with the Blowpipe 12mo,	60
Mason's Water-supply. (Considered Principally from a Sanitary Standpoint.) 3d Edition, Rewritten 8vo,	4 00
Examination of Water. (Chemical and Bacteriological.) 12mo,	1 25
Matthews's The Textile Fibres 8vo,	3 50
Meyer's Determination of Radicles in Carbon Compounds. (Tingle.) 12mo,	1 00
Miller's Manual of Assaying 12mo,	1 00
Mixter's Elementary Text-book of Chemistry 12mo,	1 50
Morgan's Outline of Theory of Solution and its Results 12mo,	1 00
Elements of Physical Chemistry 12mo,	2 00
Morse's Calculations used in Cane-sugar Factories 16mo, morocco,	1 50
Mulliken's General Method for the Identification of Pure Organic Compounds. Vol. I. Large 8vo,	5 00
O'Brine's Laboratory Guide in Chemical Analysis 8vo,	2 00
O'Driscoll's Notes on the Treatment of Gold Ores 8vo,	2 00
Ostwald's Conversations on Chemistry. Part One. (Ramsey.) 12mo,	1 50
* Penfield's Notes on Determinative Mineralogy and Record of Mineral Tests. 8vo, paper,	50
Pictet's The Alkaloids and their Chemical Constitution. (Biddle.) 8vo,	5 00
Pinner's Introduction to Organic Chemistry. (Austen.) 12mo,	1 50
Poole's Calorific Power of Fuels 8vo,	3 00
Prescott and Winslow's Elements of Water Bacteriology, with Special Reference to Sanitary Water Analysis 12mo,	1 25

* Reisig's Guide to Piece-dyeing	8vo,	25 00
Richards and Woodman's Air, Water, and Food from a Sanitary Standpoint	8vo,	2 00
Richards's Cost of Living as Modified by Sanitary Science	12mo	1 00
Cost of Food a Study in Dietaries	12mo,	1 00
* Richards and Williams's The Dietary Computer	8vo,	1 50
Ricketts and Russell's Skeleton Notes upon Inorganic Chemistry (Part I.—Non-metallic Elements.)	8vo, morocco,	75
Ricketts and Miller's Notes on Assaying	8vo,	3 00
Rideal's Sewage and the Bacterial Purification of Sewage	8vo,	3 50
Disinfection and the Preservation of Food	8vo,	4 00
Riggs's Elementary Manual for the Chemical Laboratory	8vo,	1 25
Rostoski's Serum Diagnosis. (Bolduan.)	12mo,	1 00
Ruddiman's Incompatibilities in Prescriptions	8vo,	2 00
Sabin's Industrial and Artistic Technology of Paints and Varnish	8vo,	3 00
Salkowski's Physiological and Pathological Chemistry. (Orndorff.)	8vo,	2 50
Schimpf's Text-book of Volumetric Analysis	12mo,	2 50
Essentials of Volumetric Analysis	12mo,	1 25
Spencer's Handbook for Chemists of Beet-sugar Houses	16mo, morocco,	3 00
Handbook for Sugar Manufacturers and their Chemists	16mo, morocco,	2 00
Stockbridge's Rocks and Soils	8vo,	2 50
* Tillman's Elementary Lessons in Heat	8vo,	1 50
* Descriptive General Chemistry	8vo,	3 00
Treadwell's Qualitative Analysis. (Hall.)	8vo,	3 00
Quantitative Analysis. (Hall.)	8vo,	4 00
Turneaure and Russell's Public Water-supplies	8vo,	5 00
Van Deventer's Physical Chemistry for Beginners. (Boltwood.)	12mo,	1 50
* Walke's Lectures on Explosives	8vo,	4 00
Washington's Manual of the Chemical Analysis of Rocks	8vo,	2 00
Wassermann's Immune Sera: Hæmolysins, Cytotoxins, and Precipitins. (Bolduan.)	12mo,	1 00
Wells's Laboratory Guide in Qualitative Chemical Analysis	8vo,	1 50
Short Course in Inorganic Qualitative Chemical Analysis for Engineering Students	12mo,	1 50
Whipple's Microscopy of Drinking-water	8vo,	3 50
Wiechmann's Sugar Analysis	Small 8vo,	2 50
Wilson's Cyanide Processes	12mo,	1 50
Chlorination Process	12mo,	1 50
Wulling's Elementary Course in Inorganic Pharmaceutical and Medical Chemistry	12mo,	2 00

CIVIL ENGINEERING.

BRIDGES AND ROOFS. HYDRAULICS. MATERIALS OF ENGINEERING RAILWAY ENGINEERING.

Baker's Engineers' Surveying Instruments	12mo,	3 00
Bixby's Graphical Computing Table	Paper 10½ × 24½ inches.	25
** Burr's Ancient and Modern Engineering and the Isthmian Canal. (Postage, 27 cents additional.)	8vo, net,	3 50
Comstock's Field Astronomy for Engineers	8vo,	2 50
Davis's Elevation and Stadia Tables	8vo,	1 00
Elliott's Engineering for Land Drainage	12mo,	1 50
Practical Farm Drainage	12mo,	1 00
Folwell's Sewerage. (Designing and Maintenance.)	8vo,	3 00
Freitag's Architectural Engineering. 2d Edition Rewritten	8vo	3 50
French and Ives's Stereotomy	8vo,	2 50
Goodhue's Municipal Improvements	12mo,	1 75
Goodrich's Economic Disposal of Towns' Refuse	8vo,	3 50
Gore's Elements of Geodesy	8vo,	2 50
Hayford's Text-book of Geodetic Astronomy	8vo,	3 00
Hering's Ready Reference Tables (Conversion Factors)	16mo, morocco,	2 50

Howe's Retaining Walls for Earth.............................12mo, 1 25
Johnson's (J. B.) Theory and Practice of Surveying..............Small 8vo, 4 00
Johnson's (L. J.) Statics by Algebraic and Graphic Methods.............8vo, 2 00
Laplace's Philosophical Essay on Probabilities. (Truscott and Emory.) 12mo, 2 00
Mahan's Treatise on Civil Engineering. (1873.) (Wood.).............8vo, 5 00
* Descriptive Geometry..8vo, 1 50
Merriman's Elements of Precise Surveying and Geodesy.................8vo, 2 50
Elements of Sanitary Engineering.................................8vo, 2 00
Merriman and Brooks's Handbook for Surveyors..............16mo, morocco, 2 00
Nugent's Plane Surveying..8vo 3 50
Ogden's Sewer Design..12mo, 2 00
Patton's Treatise on Civil Engineering....................8vo half leather, 7 50
Reed's Topographical Drawing and Sketching..........................4to, 5 00
Rideal's Sewage and the Bacterial Purification of Sewage.................8vo, 3 50
Siebert and Biggin's Modern Stone-cutting and Masonry.................8vo, 1 50
Smith's Manual of Topographical Drawing. (McMillan.)...............8vo, 2 50
Sondericker's Graphic Statics, with Applications to Trusses, Beams, and Arches...8vo, 2 00
Taylor and Thompson's Treatise on Concrete, Plain and Reinforced. (*In press.*)
* **Trautwine's** Civil Engineer's Pocket-book................16mo, morocco, 5 00
Wait's Engineering and Architectural Jurisprudence....................8vo, 6 00
Sheep, 6 50
Law of Operations Preliminary to Construction in Engineering and Architecture..8vo, 5 00
Sheep, 5 50
Law of Contracts...8vo, 3 00
Warren's Stereotomy—Problems in Stone-cutting......................8vo, 2 50
Webb's Problems in the Use and Adjustment of Engineering Instruments.
16mo, morocco, 1 25
* **Wheeler's** Elementary Course of Civil Engineering...................8vo, 4 00
Wilson's Topographic Surveying......................................8vo, 3 50

BRIDGES AND ROOFS.

Boller's Practical Treatise on the Construction of Iron Highway Bridges..8vo, 2 00
* Thames River Bridge.................................4to, paper, 5 00
Burr's Course on the Stresses in Bridges and Roof Trusses, Arched Ribs, and Suspension Bridges.......................................8vo, 3 50
Du Bois's Mechanics of Engineering. Vol. II.................Small 4to, 10 00
Foster's Treatise on Wooden Trestle Bridges...........................4to, 5 00
Fowler's Coffer-dam Process for Piers...............................8vo, 2 50
Ordinary Foundations...8vo, 3 50
Greene's Roof Trusses..8vo, 1 25
Bridge Trusses..8vo, 2 50
Arches in Wood, Iron, and Stone..................................8vo, 2 50
Howe's Treatise on Arches...8vo, 4 00
Design of Simple Roof-trusses in Wood and Steel..................8vo, 2 00
Johnson, Bryan, and Turneaure's Theory and Practice in the Designing of Modern Framed Structures..........................Small 4to, 10 00
Merriman and Jacoby's Text-book on Roofs and Bridges:
Part I.—Stresses in Simple Trusses...............................8vo, 2 50
Part II.—Graphic Statics...8vo, 2 50
Part III.—Bridge Design. 4th Edition, Rewritten.................8vo, 2 50
Part IV.—Higher Structures.......................................8vo, 2 50
Morison's Memphis Bridge...4to, 10 00
Waddell's De Pontibus, a Pocket-book for Bridge Engineers...16mo, morocco, 3 00
Specifications for Steel Bridges................................12mo, 1 25
Wood's Treatise on the Theory of the Construction of Bridges and Roofs.8vo, 2 00
Wright's Designing of Draw-spans:
Part I.—Plate-girder Draws......................................8vo, 2 50
Part II.—Riveted-truss and Pin-connected Long-span Draws.......8vo, 2 50
Two parts in *one* volume...8vo, 3 50

HYDRAULICS.

Bazin's Experiments upon the Contraction of the Liquid Vein Issuing from an Orifice. (Trautwine.)	8vo,	2 00
Bovey's Treatise on Hydraulics	8vo,	5 00
Church's Mechanics of Engineering	8vo,	6 00
Diagrams of Mean Velocity of Water in Open Channels	paper,	1 50
Coffin's Graphical Solution of Hydraulic Problems	16mo, morocco,	2 50
Flather's Dynamometers, and the Measurement of Power	12mo,	3 00
Folwell's Water-supply Engineering	8vo,	4 00
Frizell's Water-power	8vo,	5 00
Fuertes's Water and Public Health	12mo,	1 50
Water-filtration Works	12mo,	2 50
Ganguillet and Kutter's General Formula for the Uniform Flow of Water in Rivers and Other Channels. (Hering and Trautwine.)	8vo,	4 00
Hazen's Filtration of Public Water-supply	8vo,	3 00
Hazlehurst's Towers and Tanks for Water-works	8vo,	2 50
Herschel's 115 Experiments on the Carrying Capacity of Large, Riveted, Metal Conduits	8vo,	2 00
Mason's Water-supply. (Considered Principally from a Sanitary Standpoint.) 3d Edition, Rewritten	8vo,	4 00
Merriman's Treatise on Hydraulics. 9th Edition, Rewritten	8vo,	5 00
* Michie's Elements of Analytical Mechanics	8vo,	4 00
Schuyler's Reservoirs for Irrigation, Water-power, and Domestic Water-supply	Large 8vo,	5 00
** Thomas and Watt's Improvement of Riyers. (Post., 44 c. additional),	4to,	6 00
Turneaure and Russell's Public Water-supplies	8vo,	5 00
Wegmann's Design and Construction of Dams	4to,	5 00
Water-supply of the City of New York from 1658 to 1895	4to,	10 00
Weisbach's Hydraulics and Hydraulic Motors. (Du Bois.)	8vo,	5 00
Wilson's Manual of Irrigation Engineering	Small 8vo.	4 00
Wolff's Windmill as a Prime Mover	8vo,	3 00
Wood's Turbines	8vo,	2 50
Elements of Analytical Mechanics	8vo,	3 00

MATERIALS OF ENGINEERING.

Baker's Treatise on Masonry Construction	8vo,	5 00
Roads and Pavements	8vo,	5 00
Black's United States Public Works	Oblong 4to,	5 00
Bovey's Strength of Materials and Theory of Structures	8vo,	7 50
Burr's Elasticity and Resistance of the Materials of Engineering. 6th Edition, Rewritten	8vo,	7 50
Byrne's Highway Construction	8vo,	5 00
Inspection of the Materials and Workmanship Employed in Construction.	16mo,	3 00
Church's Mechanics of Engineering	8vo,	6 00
Du Bois's Mechanics of Engineering. Vol. I.	Small 4to,	7 50
Johnson's Materials of Construction	Large 8vo,	6 00
Fowler's Ordinary Foundations	8vo,	3 50
Keep's Cast Iron	8vo,	2 50
Lanza's Applied Mechanics	8vo,	7 50
Martens's Handbook on Testing Materials. (Henning.) 2 vols.	8vo,	7 50
Merrill's Stones for Building and Decoration	8vo,	5 00
Merriman's Text-book on the Mechanics of Materials	8vo,	4 00
Strength of Materials	12mo,	1 00
Metcalf's Steel. A Manual for Steel-users	12mo,	2 00
Patton's Practical Treatise on Foundations	8vo,	5 00
Richey's Handbook for Building Superintendents of Construction. (*In press.*)		
Rockwell's Roads and Pavements in France	12mo,	1 25

Sabin's Industrial and Artistic Technology of Paints and Varnish	8vo,	3 00
Smith's Materials of Machines	12mo,	1 00
Snow's Principal Species of Wood	8vo,	3 50
Spalding's Hydraulic Cement	12mo,	2 00
Text-book on Roads and Pavements	12mo,	2 00
Taylor and Thompson's Treatise on Concrete, Plain and Reinforced. *(In press.)*		
Thurston's Materials of Engineering. 3 Parts	8vo,	8 00
Part I.—Non-metallic Materials of Engineering and Metallurgy	8vo,	2 00
Part II.—Iron and Steel	8vo,	3 50
Part III.—A Treatise on Brasses, Bronzes, and Other Alloys and their Constituents	8vo,	2 50
Thurston's Text-book of the Materials of Construction	8vo,	5 00
Tillson's Street Pavements and Paving Materials	8vo,	4 00
Waddell's De Pontibus. (A Pocket-book for Bridge Engineers.)	16mo, mor.,	3 00
Specifications for Steel Bridges	12mo,	1 25
Wood's (De V.) Treatise on the Resistance of Materials. and an Appendix on the Preservation of Timber	8vo,	2 00
Wood's (De V.) Elements of Analytical Mechanics	8vo,	3 00
Wood's (M. P.) Rustless Coatings: Corrosion and Electrolysis of Iron and Steel	8vo.	4 00

RAILWAY ENGINEERING.

Andrews's Handbook for Street Railway Engineers	3x5 inches, morocco,	1 25
Berg's Buildings and Structures of American Railroads	4to,	5 00
Brooks's Handbook of Street Railroad Location	16mo, morocco,	1 50
Butts's Civil Engineer's Field-book	16mo, morocco,	2 50
Crandall's Transition Curve	16mo, morocco,	1 50
Railway and Other Earthwork Tables	8vo,	1 50
Dawson's "Engineering" and Electric Traction Pocket-book.	16mo, morocco,	5 00
Dredge's History of the Pennsylvania Railroad: (1879)	Paper,	5 00
* Drinker's Tunneling, Explosive Compounds, and Rock Drills,	4to, half mor.,	25 00
Fisher's Table of Cubic Yards	Cardboard,	25
Godwin's Railroad Engineers' Field-book and Explorers' Guide	16mo, mor.,	2 50
Howard's Transition Curve Field-book	16mo, morocco,	1 50
Hudson's Tables for Calculating the Cubic Contents of Excavations and Embankments	8vo,	1 00
Molitor and Beard's Manual for Resident Engineers	16mo,	1 00
Nagle's Field Manual for Railroad Engineers	16mo, morocco,	3 00
Philbrick's Field Manual for Engineers	16mo, morocco,	3 00
Searles's Field Engineering	16mo, morocco,	3 00
Railroad Spiral	16mo, morocco,	1 50
Taylor's Prismoidal Formulæ and Earthwork	8vo,	1 50
* Trautwine's Method ot Calculating the Cubic Contents of Excavations and Embankments by the Aid of Diagrams	8vo,	2 00
The Field Practice of Laying Out Circular Curves for Railroads.	12mo, morocco,	2 50
Cross-section Sheet	Paper,	25
Webb's Railroad Construction. 2d Edition, Rewritten	16mo, morocco,	5 00
Wellington's Economic Theory of the Location of Railways	Small 8vo,	5 00

DRAWING.

Barr's Kinematics of Machinery	8vo,	2 50
* Bartlett's Mechanical Drawing	8vo,	3 00
* " Abridged Ed.	8vo,	1 50
Coolidge's Manual ot Drawing	8vo, paper,	1 00
Coolidge and Freeman's Elements of General Drafting for Mechanical Engineers	Oblong 4to.	2 50
Durley's Kinematics of Machines	8vo,	4 00
Emch's Introduction to Projective Geometry and its Applications	8vo,	2 50

Hill's Text-book on Shades and Shadows, and Perspective	8vo,	2 00
Jamison's Elements of Mechanical Drawing	8vo,	2 50
Jones's Machine Design:		
Part I.—Kinematics of Machinery	8vo,	1 50
Part II.—Form, Strength, and Proportions of Parts	8vo,	3 00
MacCord's Elements of Descriptive Geometry	8vo,	3 00
Kinematics; or, Practical Mechanism	8vo,	5 00
Mechanical Drawing	4to,	4 00
Velocity Diagrams	8vo,	1 50
Mahan's Descriptive Geometry and Stone-cutting	8vo,	1 50
Industrial Drawing. (Thompson.)	8vo,	3 50
Moyer's Descriptive Geometry. (*In press.*)		
Reed's Topographical Drawing and Sketching	4to,	5 00
Reid's Course in Mechanical Drawing	8vo,	2 00
Text-book of Mechanical Drawing and Elementary Machine Design	8vo,	3 00
Robinson's Principles of Mechanism	8vo,	3 00
Schwamb and Merrill's Elements of Mechanism	8vo,	3 00
Smith's Manual of Topographical Drawing. (McMillan.)	8vo,	2 50
Warren's Elements of Plane and Solid Free-hand Geometrical Drawing	12mo,	1 00
Drafting Instruments and Operations	12mo,	1 25
Manual of Elementary Projection Drawing	12mo,	1 50
Manual of Elementary Problems in the Linear Perspective of Form and Shadow	12mo,	1 00
Plane Problems in Elementary Geometry	12mo,	1 25
Primary Geometry	12mo,	75
Elements of Descriptive Geometry, Shadows, and Perspective	8vo,	3 50
General Problems of Shades and Shadows	8vo,	3 00
Elements of Machine Construction and Drawing	8vo,	7 50
Problems, Theorems, and Examples in Descriptive Geometry	8vo,	2 50
Weisbach's Kinematics and the Power of Transmission. (Hermann and Klein.)	8vo,	5 00
Whelpley's Practical Instruction in the Art of Letter Engraving	12mo,	2 00
Wilson's (H. M.) Topographic Surveying	8vo,	3 50
Wilson's (V. T.) Free-hand Perspective	8vo,	2 50
Wilson's (V. T.) Free-hand Lettering	8vo,	1 00
Woolf's Elementary Course in Descriptive Geometry	Large 8vo,	3 00

ELECTRICITY AND PHYSICS.

Anthony and Brackett's Text-book of Physics. (Magie.)	Small 8vo,	3 00
Anthony's Lecture-notes on the Theory of Electrical Measurements	12mo,	1 00
Benjamin's History of Electricity	8vo,	3 00
Voltaic Cell	8vo,	3 00
Classen's Quantitative Chemical Analysis by Electrolysis. (Boltwood.)	8vo,	3 00
Crehore and Squier's Polarizing Photo-chronograph	8vo,	3 00
Dawson's "Engineering" and Electric Traction Pocket-book	16mo, morocco,	5 00
Dolezalek's Theory of the Lead Accumulator (Storage Battery). (Von Ende.)	12mo,	2 50
Duhem's Thermodynamics and Chemistry. (Burgess.)	8vo,	4 00
Flather's Dynamometers, and the Measurement of Power	12mo,	3 00
Gilbert's De Magnete. (Mottelay.)	8vo,	2 50
Hanchett's Alternating Currents Explained	12mo,	1 00
Hering's Ready Reference Tables (Conversion Factors)	16mo, morocco,	2 50
Holman's Precision of Measurements	8vo,	2 00
Telescopic Mirror-scale Method, Adjustments, and Tests	Large 8vo,	75
Kinzbrunner's Testing of Continuous-Current Machines	8vo,	2 00
Landauer's Spectrum Analysis. (Tingle.)	8vo,	3 00
Le Chatelier's High-temperature Measurements. (Boudouard—Burgess.)	12mo,	3 00
Löb's Electrolysis and Electrosynthesis of Organic Compounds. (Lorenz.)	12mo,	1 00

* Lyons's Treatise on Electromagnetic Phenomena. Vols. I. and II.	8vo, each,	6 00
* Michie. Elements of Wave Motion Relating to Sound and Light	8vo,	4 00
Niaudet's Elementary Treatise on Electric Batteries. (Fishback.)	12mo,	2 50
* Rosenberg's Electrical Engineering. (Haldane Gee—Kinzbrunner.)	8vo,	1 50
Ryan, Norris, and Hoxie's Electrical Machinery. Vol. I.	8vo,	2 50
Thurston's Stationary Steam-engines	8vo,	2 50
* Tillman's Elementary Lessons in Heat	8vo,	1 50
Tory and Pitcher's Manual of Laboratory Physics	Small 8vo,	2 00
Ulke's Modern Electrolytic Copper Refining	8vo,	3 00

LAW.

* Davis's Elements of Law	8vo,	2 50
* Treatise on the Military Law ot United States	8vo,	7 00
*	Sheep,	7 50
Manual for Courts-martial	16mo, morocco,	1 50
Wait's Engineering and Architectural Jurisprudence	8vo,	6 00
	Sheep,	6 50
Law of Operations Preliminary to Construction in Engineering and Architecture	8vo,	5 00
	Sheep,	5 50
Law of Contracts	8vo,	3 00
Winthrop's Abridgment of Military Law	12mo,	2 50

MANUFACTURES.

Bernadou's Smokeless Powder—Nitro-cellulose and Theory of the Cellulose Molecule	12mo,	2 50
Bolland's Iron Founder	12mo,	2 50
"The Iron Founder," Supplement	12mo,	2 50
Encyclopedia of Founding and Dictionary of Foundry Terms Used in the Practice of Moulding	12mo,	3 00
Eissler's Modern High Explosives	8vo,	4 00
Effront's Enzymes and their Applications. (Prescott.)	8vo	3 00
Fitzgerald's Boston Machinist	18mo,	1 00
Ford's Boiler Making for Boiler Makers	18mo,	1 00
Hopkins's Oil-chemists' Handbook	8vo,	3 00
Keep's Cast Iron	8vo,	2 50
Leach's The Inspection and Analysis of Food with Special Reference to State Control. (*In preparation.*)		
Matthews's The Textile Fibres	8vo,	3 50
Metcalf's Steel. A Manual for Steel-users	12mo,	2 00
Metcalfe's Cost of Manufactures—And the Administration of Workshops, Public and Private	8vo,	5 00
Meyer's Modern Locomotive Construction	4to,	10 00
Morse's Calculations used in Cane-sugar Factories	16mo, morocco,	1 50
* Reisig's Guide to Piece-dyeing	8vo,	25 00
Sabin's Industrial and Artistic Technology of Paints and Varnish	8vo,	3 00
Smith's Press-working of Metals	8vo,	3 00
Spalding's Hydraulic Cement	12mo,	2 00
Spencer's Handbook for Chemists of Beet-sugar Houses	16mo, morocco,	3 00
Handbook for Sugar Manufacturers and their Chemists	16mo morocco,	2 00
Taylor and Thompson's Treatise on Concrete, Plain and Reinforced. (*In press.*)		
Thurston's Manual of Steam-boilers, their Designs, Construction and Operation	8vo,	5 00
* Walke's Lectures on Explosives	8vo,	4 00
West's American Foundry Practice	12mo,	2 50
Moulder's Text-book	12mo,	2 50

Wolff's Windmill as a Prime Mover..................................8vo, 3 00
Woodbury's Fire Protection of Mills..................................8vo, 2 50
Wood's Rustless Coatings: Corrosion and Electrolysis of Iron and Steel...8vo, 4 00

MATHEMATICS.

Baker's Elliptic Functions..8vo, 1 50
* Bass's Elements of Differential Calculus..........................12mo, 4 00
Briggs's Elements of Plane Analytic Geometry........................12mo, 1 00
Compton's Manual of Logarithmic Computations........................12mo, 1 50
Davis's Introduction to the Logic of Algebra..........................8vo, 1 50
* Dickson's College Algebra.................................Large 12mo, 1 50
* Introduction to the Theory of Algebraic Equations ....Large 12mo, 1 25
Emch's Introduction to Projective Geometry and its Applications........8vo, 2 50
Halsted's Elements of Geometry..8vo, 1 75
Elementary Synthetic Geometry..8vo, 1 50
Rational Geometry..12mo,
* Johnson's (J. B.) Three-place Logarithmic Tables: Vest-pocket size..paper, 15
100 copies for 5 00
* Mounted on heavy cardboard, 8 × 10 inches, 25
10 copies for 2 00
Johnson's (W. W.) Elementary Treatise on Differential Calculus.. Small 8vo, 3 00
Johnson's (W. W.) Elementary Treatise on the Integral Calculus. .Small 8vo, 1 50
Johnson's (W. W.) Curve Tracing in Cartesian Co-ordinates..........12mo, 1 00
Johnson's (W. W.) Treatise on Ordinary and Partial Differential Equations.
Small 8vo, 3 50
Johnson's (W. W.) Theory of Errors and the Method of Least Squares..12mo, 1 50
* Johnson's (W. W.) Theoretical Mechanics..........................12mo, 3 00
Laplace's Philosophical Essay on Probabilities. (Truscott and Emory.) 12mo, 2 00
* Ludlow and Bass. Elements of Trigonometry and Logarithmic and Other
Tables..8vo, 3 00
Trigonometry and Tables published separately................ Each, 2 00
* Ludlow's Logarithmic and Trigonometric Tables 8vo, 1 00
Maurer's Technical Mechanics..8vo, 4 00
Merriman and Woodward's Higher Mathematics 8vo, 5 00
Merriman's Method of Least Squares....................................8vo, 2 00
Rice and Johnson's Elementary Treatise on the Differential Calculus.Sm., 8vo, 3 00
Differential and Integral Calculus. 2 vols. in one..........Small 8vo, 2 50
Wood's Elements of Co-ordinate Geometry...............................8vo, 2 00
Trigonometry: Analytical, Plane, and Spherical....................12mo, 1 00

MECHANICAL ENGINEERING.

MATERIALS OF ENGINEERING, STEAM-ENGINES AND BOILERS.

Bacon's Forge Practice...12mo, 1 50
Baldwin's Steam Heating for Buildings................................12mo, 2 50
Barr's Kinematics of Machinery..8vo, 2 50
* Bartlett's Mechanical Drawing.......................................8vo, 3 00
* " " " Abridged Ed.........................8vo, 1 50
Benjamin's Wrinkles and Recipes......................................12mo, 2 00
Carpenter's Experimental Engineering..................................8vo, 6 00
Heating and Ventilating Buildings....................................8vo, 4 00
Cary's Smoke Suppression in Plants using Bituminous Coal. (*In preparation.*)
Clerk's Gas and Oil Engine.................................Small 8vo, 4 00
Coolidge's Manual of Drawing.................................8vo, paper, 1 00
Coolidge and Freeman's Elements of General Drafting for Mechanical Engineers. ....Oblong 4to, 2 50

Cromwell's Treatise on Toothed Gearing	12mo	1 50
Treatise on Belts and Pulleys	12mo,	1 50
Durley's Kinematics of Machines	8vo,	4 00
Flather's Dynamometers and the Measurement of Power	12mo,	3 00
Rope Driving	12mo,	2 00
Gill's Gas and Fuel Analysis for Engineers	12mo,	1 25
Hall's Car Lubrication	12mo,	1 00
Hering's Ready Reference Tables (Conversion Factors)	16mo, morocco,	2 50
Hutton's The Gas Engine	8vo,	5 00
Jamison's Mechanical Drawing	8vo,	2 50
Jones's Machine Design:		
Part I.—Kinematics of Machinery	8vo,	1 50
Part II.—Form, Strength, and Proportions of Parts	8vo,	3 00
Kent's Mechanical Engineer's Pocket-book	16mo, morocco,	5 00
Kerr's Power and Power Transmission	8vo,	2 00
Leonard's Machine Shops, Tools, and Methods. (*In press.*)		
MacCord's Kinematics; or, Practical Mechanism	8vo,	5 00
Mechanical Drawing	4to,	4 00
Velocity Diagrams	8vo,	1 50
Mahan's Industrial Drawing. (Thompson.)	8vo,	3 50
Poole's Calorific Power of Fuels	8vo,	3 00
Reid's Course in Mechanical Drawing	8vo,	2 00
Text-book of Mechanical Drawing and Elementary Machine Design	8vo,	3 00
Richards's Compressed Air	12mo,	1 50
Robinson's Principles of Mechanism	8vo,	3 00
Schwamb and Merrill's Elements of Mechanism	8vo,	3 00
Smith's Press-working of Metals	8vo,	3 00
Thurston's Treatise on Friction and Lost Work in Machinery and Mill Work	8vo,	3 00
Animal as a Machine and Prime Motor, and the Laws of Energetics	12mo,	1 00
Warren's Elements of Machine Construction and Drawing	8vo,	7 50
Weisbach's Kinematics and the Power of Transmission. Herrmann—Klein.)	8vo,	5 00
Machinery of Transmission and Governors. (Herrmann—Klein.)	8vo,	5 00
Hydraulics and Hydraulic Motors. (Du Bois.)	8vo,	5 00
Wolff's Windmill as a Prime Mover	8vo,	3 00
Wood's Turbines	8vo,	2 50

MATERIALS OF ENGINEERING.

Bovey's Strength of Materials and Theory of Structures	8vo,	7 50
Burr's Elasticity and Resistance of the Materials of Engineering. 6th Edition Reset	8vo,	7 50
Church's Mechanics of Engineering	8vo,	6 00
Johnson's Materials of Construction	Large 8vo,	6 00
Keep's Cast Iron	8vo,	2 50
Lanza's Applied Mechanics	8vo,	7 50
Martens's Handbook on Testing Materials. (Henning.)	8vo,	7 50
Merriman's Text-book on the Mechanics of Materials	8vo,	4 00
Strength of Materials	12mo,	1 00
Metcalf's Steel. A Manual for Steel-users	12mo	2 00
Sabin's Industrial and Artistic Technology of Paints and Varnish	8vo,	3 00
Smith's Materials of Machines	12mo,	1 00
Thurston's Materials of Engineering	3 vols., 8vo,	8 00
Part II.—Iron and Steel	8vo,	3 50
Part III.—A Treatise on Brasses, Bronzes, and Other Alloys and their Constituents	8vo	2 50
Text-book of the Materials of Construction	8vo,	5 00

Wood's (De V.) Treatise on the Resistance of Materials and an Appendix on the Preservation of Timber	8vo,	2 00
Wood's (De V.) Elements of Analytical Mechanics	8vo,	3 00
Wood's (M. P.) Rustless Coatings: Corrosion and Electrolysis of Iron and Steel.	8vo,	4 00

STEAM-ENGINES AND BOILERS.

Carnot's Reflections on the Motive Power of Heat. (Thurston.)	12mo,	1 50
Dawson's "Engineering" and Electric Traction Pocket-book	16mo, mor.,	5 00
Ford's Boiler Making for Boiler Makers	18mo,	1 00
Goss's Locomotive Sparks	8vo,	2 00
Hemenway's Indicator Practice and Steam-engine Economy	12mo,	2 00
Hutton's Mechanical Engineering of Power Plants	8vo,	5 00
Heat and Heat-engines	8vo,	5 00
Kent's Steam-boiler Economy	8vo,	4 00
Kneass's Practice and Theory of the Injector	8vo,	1 50
MacCord's Slide-valves	8vo,	2 00
Meyer's Modern Locomotive Construction	4to,	10 00
Peabody's Manual of the Steam-engine Indicator	12mo,	1 50
Tables of the Properties of Saturated Steam and Other Vapors	8vo,	1 00
Thermodynamics of the Steam-engine and Other Heat-engines	8vo,	5 00
Valve-gears for Steam-engines	8vo,	2 50
Peabody and Miller's Steam-boilers	8vo,	4 00
Pray's Twenty Years with the Indicator	Large 8vo,	2 50
Pupin's Thermodynamics of Reversible Cycles in Gases and Saturated Vapors. (Osterberg.)	12mo,	1 25
Reagan's Locomotives: Simple, Compound, and Electric	12mo,	2 50
Rontgen's Principles of Thermodynamics. (Du Bois.)	8vo,	5 00
Sinclair's Locomotive Engine Running and Management	12mo,	2 00
Smart's Handbook of Engineering Laboratory Practice	12mo,	2 50
Snow's Steam-boiler Practice	8vo,	3 00
Spangler's Valve-gears	8vo,	2 50
Notes on Thermodynamics	12mo,	1 00
Spangler, Greene, and Marshall's Elements of Steam-engineering	8vo,	3 00
Thurston's Handy Tables	8vo,	1 50
Manual of the Steam-engine	2 vols. 8vo,	10 00
Part I.—History, Structuce, and Theory	8vo,	6 00
Part II.—Design, Construction, and Operation	8vo,	6 00
Handbook of Engine and Boiler Trials, and the Use of the Indicator and the Prony Brake	8vo,	5 00
Stationary Steam-engines	8vo,	2 50
Steam-boiler Explosions in Theory and in Practice	12mo,	1 50
Manual of Steam-boilers, Their Designs, Construction, and Operation	8vo,	5 00
Weisbach's Heat, Steam, and Steam-engines. (Du Bois.)	8vo,	5 00
Whitham's Steam-engine Design	8vo,	5 00
Wilson's Treatise on Steam-boilers. (Flather.)	16mo,	2 50
Wood's Thermodynamics Heat Motors, and Refrigerating Machines	8vo,	4 00

MECHANICS AND MACHINERY.

Barr's Kinematics of Machinery	8vo,	2 50
Bovey's Strength of Materials and Theory of Structures	8vo,	7 50
Chase's The Art of Pattern-making	12mo,	2 50
Church's Mechanics of Engineering	8vo,	6 00

Church's Notes and Examples in Mechanics	8vo,	2 00
Compton's First Lessons in Metal-working	12mo,	1 50
Compton and De Groodt's The Speed Lathe	12mo,	1 50
Cromwell's Treatise on Toothed Gearing	12mo,	1 50
Treatise on Belts and Pulleys	12mo,	1 50
Dana's Text-book of Elementary Mechanics for the Use of Colleges and Schools	12mo,	1 50
Dingey's Machinery Pattern Making	12mo,	2 00
Dredge's Record of the Transportation Exhibits Building of the World's Columbian Exposition of 1893	4to half morocco,	5 00
Du Bois's Elementary Principles of Mechanics:		
Vol. I.—Kinematics	8vo,	3 50
Vol. II.—Statics	8vo,	4 00
Vol. III.—Kinetics	8vo,	3 50
Mechanics of Engineering. Vol. I.	Small 4to,	7 50
Vol. II.	Small 4to,	10 00
Durley's Kinematics of Machines	8vo,	4 00
Fitzgerald's Boston Machinist	16mo,	1 00
Flather's Dynamometers, and the Measurement of Power	12mo,	3 00
Rope Driving	12mo,	2 00
Goss's Locomotive Sparks	8vo,	2 00
Hall's Car Lubrication	12mo,	1 00
Holly's Art of Saw Filing	18mo,	75
* Johnson's (W. W.) Theoretical Mechanics	12mo,	3 00
Johnson's (L. J.) Statics by Graphic and Algebraic Methods	8vo,	2 00
Jones's Machine Design:		
Part I.—Kinematics of Machinery	8vo,	1 50
Part II.—Form, Strength, and Proportions of Parts	8vo,	3 00
Kerr's Power and Power Transmission	8vo,	2 00
Lanza's Applied Mechanics	8vo,	7 50
Leonard's Machine Shops, Tools, and Methods. (*In press.*)		
MacCord's Kinematics; or, Practical Mechanism	8vo,	5 00
Velocity Diagrams	8vo,	1 50
Maurer's Technical Mechanics	8vo,	4 00
Merriman's Text-book on the Mechanics of Materials	8vo,	4 00
Elements of Mechanics	12mo,	1 00
* Michie's Elements of Analytical Mechanics	8vo	4 00
Reagan's Locomotives: Simple, Compound, and Electric	12mo,	2 50
Reid's Course in Mechanical Drawing	8vo,	2 00
Text-book of Mechanical Drawing and Elementary Machine Design	8vo,	3 00
Richards's Compressed Air	12mo,	1 50
Robinson's Principles of Mechanism	8vo,	3 00
Ryan, Norris, and Hoxie's Electrical Machinery. Vol. I	8vo,	2 50
Schwamb and Merrill's Elements of Mechanism	8vo,	3 00
Sinclair's Locomotive-engine Running and Management	12mo,	2 00
Smith's Press-working of Metals	8vo,	3 00
Materials of Machines	12mo,	1 00
Spangler, Greene, and Marshall's Elements of Steam-engineering	8vo,	3 00
Thurston's Treatise on Friction and Lost Work in Machinery and Mill Work	8vo,	3 00
Animal as a Machine and Prime Motor, and the Laws of Energetics	12mo,	1 00
Warren's Elements of Machine Construction and Drawing	8vo,	7 50
Weisbach's Kinematics and the Power of Transmission. (Herrmann—Klein.)	8vo,	5 00
Machinery of Transmission and Governors. (Herrmann—Klein.)	8vo,	5 00
Wood's Elements of Analytical Mechanics	8vo,	3 00
Principles of Elementary Mechanics	12mo	1 25
Turbines	8vo,	2 50
The World's Columbian Exposition of 1893	4to,	1 00

METALLURGY.

Egleston's Metallurgy of Silver, Gold, and Mercury:		
Vol. I.—Silver	8vo,	7 50
Vol. II.—Gold and Mercury	8vo,	7 50
** Iles's Lead-smelting. (Postage 9 cents additional.)	12mo,	2 50
Keep's Cast Iron	8vo,	2 50
Kunhardt's Practice of Ore Dressing in Europe	8vo,	1 50
Le Chatelier's High-temperature Measurements. (Boudouard—Burgess.)	12mo,	3 00
Metcalf's Steel. A Manual for Steel-users	12mo,	2 00
Smith's Materials of Machines	12mo,	1 00
Thurston's Materials of Engineering. In Three Parts	8vo,	8 00
Part II.—Iron and Steel	8vo,	3 50
Part III.—A Treatise on Brasses, Bronzes, and Other Alloys and their Constituents	8vo,	2 50
Ulke's Modern Electrolytic Copper Refining	8vo,	3 00

MINERALOGY.

Barringer's Description of Minerals of Commercial Value.	Oblong, morocco,	2 50
Boyd's Resources of Southwest Virginia	8vo,	3 00
Map of Southwest Virginia	Pocket-book form,	2 00
Brush's Manual of Determinative Mineralogy. (Penfield.)	8vo,	4 00
Chester's Catalogue of Minerals	8vo, paper, Cloth,	1 00 1 25
Dictionary of the Names of Minerals	8vo,	3 50
Dana's System of Mineralogy	Large 8vo, half leather,	12 50
First Appendix to Dana's New "System of Mineralogy."	Large 8vo,	1 00
Text-book of Mineralogy	8vo,	4 00
Minerals and How to Study Them	12mo,	1 50
Catalogue of American Localities of Minerals	Large 8vo,	1 00
Manual of Mineralogy and Petrography	12mo,	2 00
Douglas's Untechnical Addresses on Technical Subjects	12mo,	1 00
Eakle's Mineral Tables	8vo,	1 25
Egleston's Catalogue of Minerals and Synonyms	8vo,	2 50
Hussak's The Determination of Rock-forming Minerals. (Smith.)	Small 8vo,	2 00
Merrill's Non-metallic Minerals: Their Occurrence and Uses	8vo,	4 00
* Penfield's Notes on Determinative Mineralogy and Record of Mineral Tests.	8vo, paper,	0 50
Rosenbusch's Microscopical Physiography of the Rock-making Minerals. (Iddings.)	8vo,	5 00
* Tillman's Text-book of Important Minerals and Docks	8vo,	2 00
Williams's Manual of Lithology	8vo,	3 00

MINING.

Beard's Ventilation of Mines	12mo,	2 50
Boyd's Resources of Southwest Virginia	8vo,	3 00
Map of Southwest Virginia	Pocket-book form,	2 00
Douglas's Untechnical Addresses on Technical Subjects	12mo,	1 00
* Drinker's Tunneling, Explosive Compounds, and Rock Drills.	4to, half morocco,	25 00
Eissler's Modern High Explosives	8vo,	4 00
Fowler's Sewage Works Analyses	12mo,	2 00
Goodyear's Coal-mines of the Western Coast of the United States	12mo,	2 50
Ihlseng's Manual of Mining	8vo,	4 00
** Iles's Lead-smelting. (Postage 9c. additional.)	12mo,	2 50
Kunhardt's Practice of Ore Dressing in Europe	8vo,	1 50
O'Driscoll's Notes on the Treatment of Gold Ores	8vo,	2 00
* Walke's Lectures on Explosives	8vo,	4 00
Wilson's Cyanide Processes	12mo,	1 50
Chlorination Process	12mo,	1 50

Wilson's Hydraulic and Placer Mining	12mo,	2 00
Treatise on Practical and Theoretical Mine Ventilation	12mo,	1 25

SANITARY SCIENCE.

Folwell's Sewerage. (Designing, Construction, and Maintenance.)	8vo,	3 00
Water-supply Engineering	8vo,	4 00
Fuertes's Water and Public Health	12mo,	1 50
Water-filtration Works	12mo,	2 50
Gerhard's Guide to Sanitary House-inspection	16mo,	1 00
Goodrich's Economical Disposal of Town's Refuse	Demy 8vo,	3 50
Hazen's Filtration of Public Water-supplies	8vo,	3 00
Leach's The Inspection and Analysis of Food with Special Reference to State Control	8vo,	7 50
Mason's Water-supply. (Considered Principally from a Sanitary Standpoint.) 3d Edition, Rewritten	8vo,	4 00
Examination of Water. (Chemical and Bacteriological.)	12mo,	1 25
Merriman's Elements of Sanitary Engineering	8vo,	2 00
Ogden's Sewer Design	12mo,	2 00
Prescott and Winslow's Elements of Water Bacteriology, with Special Reference to Sanitary Water Analysis	12mo,	1 25
* Price's Handbook on Sanitation	12mo,	1 50
Richards's Cost of Food. A Study in Dietaries	12mo,	1 00
Cost of Living as Modified by Sanitary Science	12mo,	1 00
Richards and Woodman's Air, Water, and Food from a Sanitary Standpoint	8vo,	2 00
* Richards and Williams's The Dietary Computer.	8vo,	1 50
Rideal's Sewage and Bacterial Purification of Sewage.	8vo,	3 50
Turneaure and Russell's Public Water-supplies	8vo,	5 00
Von Behring's Suppression of Tuberculosis. (Bolduan.)	12mo,	1 00
Whipple's Microscopy of Drinking-water	8vo,	3 50
Woodhull's Notes and Military Hygiene	16mo,	1 50

MISCELLANEOUS.

De Fursac's Manual of Psychiatry. (Rosanoff.)	12mo,	2 50
Emmons's Geological Guide-book of the Rocky Mountain Excursion of the International Congress of Geologists	Large 8vo,	1 50
Ferrel's Popular Treatise on the Winds	8vo,	4 00
Haines's American Railway Management.	12mo,	2 50
Mott's Composition, Digestibility, and Nutritive Value of Food. Mounted chart.		1 25
Fallacy of the Present Theory of Sound	16mo,	1 00
Ricketts's History of Rensselaer Polytechnic Institute, 1824-1894.	Small 8vo,	3 00
Rostoski's Serum Diagnosis. (Bolduan.)	12mo,	1 00
Rotherham's Emphasized New Testament	Large 8vo,	2 00
Steel's Treatise on the Diseases of the Dog	8vo,	3 50
Totten's Important Question in Metrology	8vo,	2 50
The World's Columbian Exposition of 1893	4to,	1 00
Von Behring's Suppression of Tuberculosis. (Bolduan.)	12mo,	1 00
Worcester and Atkinson. Small Hospitals, Establishment and Maintenance and Suggestions for Hospital Architecture, with Plans for a Small Hospital	12mo,	1 25

HEBREW AND CHALDEE TEXT-BOOKS.

Green's Grammar of the Hebrew Language	8vo,	3 00
Elementary Hebrew Grammar	12mo,	1 25
Hebrew Chrestomathy	8vo,	2 00
Gesenius's Hebrew and Chaldee Lexicon to the Old Testament Scriptures. (Tregelles.)	Small 4to, half morocco,	5 00
Letteris's Hebrew Bible	8vo,	2 25

CENTRAL COLLECTION

46
NT
4m